

Binomial Theorem when n is negative or fraction:

When n is negative or fraction and $|x| < 1$ then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Where the general term of binomial expansion is

$$T_{r+1} = \frac{n(n-1)(n-2)\dots(n-(r-1))}{r!}x^r$$

Question # 1

Expand the following upto 4 times, taking the values of x such that the expansion in each case is valid.

- (i) $(1-x)^{\frac{1}{2}}$ (ii) $(1+2x)^{-1}$ (iii) $(1+x)^{\frac{1}{3}}$ (iv) $(4-3x)^{\frac{1}{2}}$
- (v) $(8-2x)^{-1}$ (vi) $(2-3x)^{-2}$ (vii) $\frac{(1-x)^{-1}}{(1+x)^2}$ (viii) $\frac{\sqrt{(1+2x)}}{(1-x)}$
- (ix) $\frac{(4+2x)^{\frac{1}{2}}}{(2-x)}$ (x) $(1+x-2x^2)^{\frac{1}{2}}$ (xi) $(1-2x+3x^2)^{\frac{1}{2}}$

Solution

$$\begin{aligned} \text{(i)} \quad (1-x)^{\frac{1}{2}} &= 1 + \frac{1}{2}(-x) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(-x)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3 + \dots \\ &= 1 - \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3 \cdot 2}(-x^3) + \dots \\ &= 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 + \dots \end{aligned}$$

(ii) *Do yourself as above*

(iii) *Do yourself as above*

$$\text{(iv)} \quad (4-3x)^{\frac{1}{2}} = \left[4\left(1-\frac{3x}{4}\right)\right]^{\frac{1}{2}} = (4)^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{1}{2}} = 2\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$$

$$= 2 \left[1 + \frac{1}{2}\left(-\frac{3x}{4}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{3x}{4}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{3x}{4}\right)^3 + \dots \right]$$

$$\begin{aligned}
 &= 2 \left[1 - \frac{3x}{8} + \frac{1}{2} \left(\frac{-1}{2} \right) \left(\frac{9x^2}{16} \right) + \frac{1}{2} \left(\frac{-1}{2} \right) \left(\frac{-3}{2} \right) \left(\frac{-27x^3}{64} \right) + \dots \right] \\
 &= 2 \left[1 - \frac{3x}{8} - \frac{1}{8} \left(\frac{9x^2}{16} \right) - \frac{1}{16} \left(\frac{27x^3}{64} \right) + \dots \right] \\
 &= 2 \left[1 - \frac{3x}{8} - \frac{9x^2}{128} - \frac{27x^3}{1024} + \dots \right] \\
 &= 2 - \frac{3x}{4} - \frac{9x^2}{64} - \frac{27x^3}{512} + \dots
 \end{aligned}$$

(v) $(8 - 2x)^{\frac{1}{2}} = (8)^{-1} \left(1 - \frac{2x}{8} \right)^{-1} = \frac{1}{8} \left(1 - \frac{x}{4} \right)^{-1}$ *Now do yourself*

(vi) *Do yourself*

(vii) $\frac{(1-x)^{-1}}{(1+x)^2} = (1-x)^{-1} (1+x)^{-2}$

$$\begin{aligned}
 &= \left(1 + (-1)(-x) + \frac{(-1)(-1-1)}{2!} (-x)^2 + \frac{(-1)(-1-1)(-1-2)}{3!} (-x)^3 + \dots \right) \\
 &\times \left(1 + (-2)(x) + \frac{(-2)(-2-1)}{2!} (x)^2 + \frac{(-2)(-2-1)(-2-2)}{3!} (x)^3 + \dots \right) \\
 &= \left(1 + x + \frac{(-1)(-2)}{2} (x^2) + \frac{(-1)(-2)(-3)}{3 \cdot 2} (-x^3) + \dots \right) \\
 &\times \left(1 - 2x + \frac{(-2)(-3)}{2} (x)^2 + \frac{(-2)(-3)(-4)}{3 \cdot 2} (x)^3 + \dots \right) \\
 &= (1 + x + x^2 + x^3 + \dots) \times (1 - 2x + 3x^2 - 4x^3 + \dots) \\
 &= 1 + (x - 2x) + (x^2 - 2x^2 + 3x^2) + (x^3 - 2x^3 + 3x^3 - 4x^3) + \dots \\
 &= 1 - x + 2x^2 - 2x^3 + \dots
 \end{aligned}$$

(viii) *Do yourself as above*

(ix) $\frac{(4+2x)^{\frac{1}{2}}}{2-x} = (4+2x)^{\frac{1}{2}} (2-x)^{-1} = (4)^{\frac{1}{2}} \left(1 + \frac{2x}{4} \right)^{\frac{1}{2}} (2)^{-1} \left(1 - \frac{x}{2} \right)^{-1}$

$$= (4)^{\frac{1}{2}} \left(1 + \frac{x}{2} \right)^{\frac{1}{2}} (2)^{-1} \left(1 - \frac{x}{2} \right)^{-1} = 2 \left(1 + \frac{x}{2} \right)^{\frac{1}{2}} \frac{1}{2} \left(1 - \frac{x}{2} \right)^{-1} = \left(1 + \frac{x}{2} \right)^{\frac{1}{2}} \left(1 - \frac{x}{2} \right)^{-1}$$

$$\begin{aligned}
 &= \left(1 + \frac{x}{2}\right)^{\frac{1}{2}} \left(1 - \frac{x}{2}\right)^{-1} \\
 &= \left(1 + \frac{1}{2}\left(\frac{x}{2}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3 + \dots\right) \\
 &\times \left(1 + (-1)\left(-\frac{x}{2}\right) + \frac{(-1)(-1-1)}{2!}\left(-\frac{x}{2}\right)^2 + \frac{(-1)(-1-1)(-1-2)}{3!}\left(-\frac{x}{2}\right)^3 + \dots\right) \\
 &= \left(1 + \frac{x}{4} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}\left(\frac{x^2}{4}\right) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3 \cdot 2}\left(\frac{x^3}{8}\right) + \dots\right) \\
 &\times \left(1 + \frac{x}{2} + \frac{(-1)(-2)}{2}\left(\frac{x^2}{4}\right) + \frac{(-1)(-2)(-3)}{3 \cdot 2}\left(-\frac{x^3}{8}\right) + \dots\right) \\
 &= \left(1 + \frac{x}{4} - \frac{x^2}{32} + \frac{x^3}{128} + \dots\right) \times \left(1 + \frac{x}{2} + \frac{x^2}{4} + \frac{x^3}{8} + \dots\right) \\
 &= 1 + \left(\frac{x}{4} + \frac{x}{2}\right) + \left(-\frac{x^2}{32} + \frac{x^2}{8} + \frac{x^2}{4}\right) + \left(\frac{x^3}{128} - \frac{x^3}{64} + \frac{x^3}{16} + \frac{x^3}{8}\right) + \dots \\
 &= 1 + \frac{3x}{4} + \frac{11x^2}{32} + \frac{23x^3}{128} + \dots
 \end{aligned}$$

(x) $(1 + x - 2x^2)^{\frac{1}{2}} = (1 + (x - 2x^2))^{\frac{1}{2}}$

$$\begin{aligned}
 &= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}(x - 2x^2)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(x - 2x^2)^3 + \dots \\
 &= 1 + \frac{1}{2}(x - 2x^2) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}(x^2 - 4x^3 + 4x^4) + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3 \cdot 2} \\
 &\quad (x^3 + 3(x)^2(-2x^2) + 3(x)(-2x^2)^2 - (2x^2)^3) + \dots \\
 &= 1 + \frac{1}{2}(x - 2x^2) - \frac{1}{8}(x^2 - 4x^3 + 4x^4) + \frac{1}{16}(x^3 - 6x^4 + 12x^5 - 8x^6) + \dots \\
 &= 1 + \frac{1}{2}x - \frac{2}{2}x^2 - \frac{1}{8}x^2 - \frac{4}{8}x^3 + \frac{4}{8}x^4 + \frac{1}{16}x^3 - \frac{6}{16}x^4 + \frac{12}{16}x^5 - \frac{8}{16}x^6 + \dots \\
 &= 1 + \frac{1}{2}x - x^2 - \frac{1}{8}x^2 - \frac{1}{2}x^3 + \frac{1}{2}x^4 + \frac{1}{16}x^3 - \frac{3}{8}x^4 + \frac{3}{4}x^5 - \frac{1}{8}x^6 + \dots \\
 &= 1 + \frac{1}{2}x - \frac{9}{8}x^2 - \frac{9}{16}x^3 + \dots
 \end{aligned}$$

(xi)

Do yourself as above

Question # 2

Use the Binomial theorem find the value of the following to three places of decimals.

- | | | | |
|----------------------------------|-----------------------------|---------------------------------|----------------------------------|
| (i) $\sqrt{99}$ | (ii) $(0.98)^{\frac{1}{2}}$ | (iii) $(1.03)^{\frac{1}{3}}$ | (iv) $\sqrt[3]{65}$ |
| (v) $\sqrt[4]{17}$ | (vi) $\sqrt[5]{31}$ | (vii) $\frac{1}{\sqrt[3]{998}}$ | (viii) $\frac{1}{\sqrt[5]{252}}$ |
| (ix) $\frac{\sqrt{7}}{\sqrt{8}}$ | (x) $(0.998)^{\frac{1}{3}}$ | (xi) $\frac{1}{\sqrt[6]{486}}$ | (xii) $(1280)^{\frac{1}{4}}$ |

Solution

$$\begin{aligned}
 \text{(i)} \quad \sqrt{99} &= (99)^{\frac{1}{2}} = (100 - 1)^{\frac{1}{2}} = (100)^{\frac{1}{2}} \left(1 - \frac{1}{100} \right)^{\frac{1}{2}} \\
 &= 10 \left(1 + \frac{1}{2} \left(-\frac{1}{100} \right) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} \left(-\frac{1}{100} \right)^2 + \dots \right) \\
 &= 10 \left(1 - \frac{1}{200} + \frac{\frac{1}{2} \left(-\frac{1}{2} \right)}{2} \left(\frac{1}{10000} \right) + \dots \right) \\
 &= 10 \left(1 - 0.005 - \frac{1}{8} (0.0001) + \dots \right) \\
 &= 10(1 - 0.005 - 0.0000125 + \dots) \\
 &\approx 10(0.9949875) = 9.949875 \\
 &\approx 9.950
 \end{aligned}$$

$$\text{(ii)} \quad (0.98)^{\frac{1}{2}} = (1 - 0.02)^{\frac{1}{2}} \quad \text{Now do yourself}$$

$$\text{(iii)} \quad (1.03)^{\frac{1}{3}} = (1 + 0.03)^{\frac{1}{3}} \quad \text{Now do yourself}$$

$$\text{(iv)} \quad \sqrt[3]{65} = (65)^{\frac{1}{3}} = (64 - 1)^{\frac{1}{3}} = (64)^{\frac{1}{3}} \left(1 - \frac{1}{64} \right)^{\frac{1}{3}} \quad \text{Now do yourself}$$

$$\text{(v)} \quad \sqrt[4]{17} = (17)^{\frac{1}{4}} = (16 - 1)^{\frac{1}{4}} = (16)^{\frac{1}{4}} \left(1 - \frac{1}{16} \right)^{\frac{1}{4}} \quad \text{Now do yourself}$$

$$\text{(vi)} \quad \sqrt[5]{31} = (31)^{\frac{1}{5}} = (32 - 1)^{\frac{1}{5}} = (32)^{\frac{1}{5}} \left(1 - \frac{1}{32} \right)^{\frac{1}{5}} \quad \text{Now do yourself}$$

$$\begin{aligned}
\text{(vii)} \quad \frac{1}{\sqrt[3]{998}} &= \frac{1}{(998)^{\frac{1}{3}}} = (998)^{-\frac{1}{3}} = (1000 - 2)^{-\frac{1}{3}} = (1000)^{-\frac{1}{3}} \left(1 - \frac{2}{1000}\right)^{-\frac{1}{3}} \\
&= (10^3)^{-\frac{1}{3}} \left(1 - \frac{1}{500}\right)^{-\frac{1}{3}} \\
&= \left(\frac{1}{10}\right) \left(1 + \left(-\frac{1}{3}\right)\left(-\frac{1}{500}\right) + \frac{-\frac{1}{3}\left(-\frac{1}{3}-1\right)}{2!} \left(-\frac{1}{500}\right)^2 + \dots\right) \\
&= \left(\frac{1}{10}\right) \left(1 + \left(\frac{1}{1500}\right) + \frac{-\frac{1}{3}\left(-\frac{4}{3}\right)}{2} \left(\frac{1}{250000}\right) + \dots\right) \\
&= \left(\frac{1}{10}\right) \left(1 + (0.0006667) + \frac{2}{9}(0.000004) + \dots\right) \\
&= \left(\frac{1}{10}\right) (1 + 0.0006667 + 0.00000089 + \dots) \\
&\approx \left(\frac{1}{10}\right) (1.00066759) = 0.100066759 \approx 0.100 \quad \text{Answer}
\end{aligned}$$

$$\begin{aligned}
\text{(viii)} \quad \frac{1}{\sqrt[5]{252}} &= \frac{1}{(252)^{\frac{1}{5}}} = (252)^{-\frac{1}{5}} = (243 + 9)^{-\frac{1}{5}} = (243)^{-\frac{1}{5}} \left(1 + \frac{9}{243}\right)^{-\frac{1}{5}} \\
&= (3^5)^{-\frac{1}{5}} \left(1 + \frac{1}{27}\right)^{-\frac{1}{5}} \quad \text{Now do yourself as above}
\end{aligned}$$

$$\begin{aligned}
\text{(ix)} \quad \frac{\sqrt{7}}{\sqrt{8}} &= \sqrt{\frac{7}{8}} = \left(\frac{7}{8}\right)^{\frac{1}{2}} = \left(1 - \frac{1}{8}\right)^{\frac{1}{2}} \\
&= 1 + \frac{1}{2}\left(-\frac{1}{8}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} \left(-\frac{1}{8}\right)^2 + \dots \\
&= 1 - \frac{1}{16} + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2} \left(\frac{1}{64}\right) + \dots \\
&= 1 - \frac{1}{16} - \frac{1}{8}\left(\frac{1}{64}\right) + \dots \\
&= 1 - \frac{1}{16} - \frac{1}{512} + \dots \\
&= 1 - 0.0625 - 0.00195 + \dots \\
&\approx 0.93555 \approx 0.936 \quad \text{Answer}
\end{aligned}$$

(x) $(0.998)^{\frac{1}{3}} = (1 - 0.002)^{\frac{1}{3}}$ *Now do yourself as above*

(xi)
$$\frac{1}{\sqrt[6]{486}} = \frac{1}{(486)^{\frac{1}{6}}} = (486)^{-\frac{1}{6}} = (729 - 243)^{-\frac{1}{6}} = (729)^{-\frac{1}{6}} \left(1 - \frac{243}{729}\right)^{-\frac{1}{6}}$$

$$= (3^6)^{-\frac{1}{6}} \left(1 - \frac{1}{3}\right)^{-\frac{1}{6}} \quad \text{Now do yourself}$$

(xii)
$$(1280)^{\frac{1}{4}} = (1296 - 16)^{\frac{1}{4}} = (1296)^{\frac{1}{4}} \left(1 - \frac{16}{1296}\right)^{\frac{1}{4}} = (6^4)^{\frac{1}{4}} \left(1 - \frac{1}{81}\right)^{\frac{1}{4}}$$
Now do yourself

Question # 3

Find the coefficient of x^n in the expansion of

(i) $\frac{1+x^2}{(1+x)^2}$

(ii) $\frac{(1+x)^2}{(1-x)^2}$

(iii) $\frac{(1+x)^3}{(1-x)^2}$

(iv) $\frac{(1+x)^2}{(1-x)^3}$

(v) $(1-x+x^2-x^3+\dots)^2$

Solution

(i)
$$\frac{1+x^2}{(1+x)^2} = (1+x^2)(1+x)^{-2}$$

$$= (1+x^2) \left(1 + (-2)(x) + \frac{(-2)(-2-1)}{2!}(x)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}(x)^3 + \dots \right)$$

$$= (1+x^2) \left(1 - 2x + \frac{(-2)(-3)}{2}x^2 + \frac{(-2)(-3)(-4)}{3 \cdot 2}x^3 + \dots \right)$$

$$= (1+x^2)(1 - 2x + 3x^2 - 4x^3 + \dots)$$

$$= (1+x^2)(1 + (-1)2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \dots)$$

Following in this way we can write

$$\frac{1+x^2}{(1+x)^2} = (1+x^2)(1 + (-1)2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \dots + (-1)^{n-2}(n-1)x^{n-2} + (-1)^{n-1}(n)x^{n-1} + (-1)^n(n+1)x^n + \dots)$$

So taking only terms involving x^n we get

$$\begin{aligned} & (-1)^n(n+1)x^n + (-1)^{n-2}(n-1)x^n \\ &= (-1)^n(n+1)x^n + (-1)^n(-1)^{-2}(n-1)x^n \\ &= (-1)^n(n+1)x^n + (-1)^n(n-1)x^n \qquad \because (-1)^{-2} = 1 \\ &= (n+1+n-1)(-1)^n x^n = (2n)(-1)^n x^n \end{aligned}$$

Thus the coefficient of term involving x^n is $(2n)(-1)^n$

(ii)

Hint:

After solving you will get

$$\frac{(1+x)^2}{(1-x)^2} = (1+2x+x^2)(1+2x+3x^2+4x^3+\dots+(n-1)x^{n-2}+(n)x^{n-1}+(n+1)x^n+\dots)$$

Do yourself as above

$$\begin{aligned} \text{(iii)} \quad \frac{(1+x)^3}{(1-x)^2} &= (1+x)^3(1-x)^{-2} \\ &= (1+x)^3 \left(1 + (-2)(-x) + \frac{(-2)(-2-1)}{2!}(-x)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}(-x)^3 + \dots \right) \\ &= (1+x)^3 \left(1 + 2x + \frac{(-2)(-3)}{2}(x)^2 + \frac{(-2)(-3)(-4)}{3 \cdot 2}(-x^3) + \dots \right) \\ &= (1+3x+3x^2+x^3)(1+2x+3x^2+4x^3+\dots) \end{aligned}$$

Following in this way we can write

$$\begin{aligned} \frac{(1+x)^3}{(1-x)^2} &= (1+3x+3x^2+x^3)(1+2x+3x^2+4x^3+\dots+(n-2)x^{n-3}+(n-1)x^{n-2} \\ &\quad + (n)x^{n-1}+(n+1)x^n+\dots) \end{aligned}$$

So taking only terms involving x^n we have term

$$\begin{aligned} &(n+1)x^n + 3(n)x^n + 3(n-1)x^n + (n-2)x^n \\ &= ((n+1) + 3(n) + 3(n-1) + (n-2))x^n \\ &= (n+1+3n+3n-3+n-2)x^n \\ &= (8n-4)x^n \end{aligned}$$

Thus the coefficient of term involving x^n is $(8n-4)$.

$$\begin{aligned} \text{(iv)} \quad \frac{(1+x)^2}{(1-x)^3} &= (1+x)^2(1-x)^{-3} \\ &= (1+x)^2 \left(1 + (-3)(-x) + \frac{(-3)(-3-1)}{2!}(-x)^2 + \frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3 + \dots \right) \\ &= (1+x)^2 \left(1 + (-3)(-x) + \frac{(-3)(-4)}{2}(-x)^2 + \frac{(-3)(-4)(-5)}{3 \cdot 2}(-x)^3 + \dots \right) \\ &= (1+2x+x^2) \left(1 + 3x + \frac{(3)(4)}{2}(x^2) + \frac{(4)(5)}{2}(x^3) + \dots \right) \\ &= (1+2x+x^2) \left(1 + \frac{(2)(3)}{2}x + \frac{(3)(4)}{2}x^2 + \frac{(4)(5)}{2}x^3 + \dots \right) \end{aligned}$$

Following in this way we can write

$$\frac{(1+x)^2}{(1-x)^3} = (1+2x+x^2) \left(1 + \frac{(2)(3)}{2}x + \frac{(3)(4)}{2}x^2 + \frac{(4)(5)}{2}x^3 + \dots \right. \\ \left. + \frac{(n-1)(n)}{2}x^{n-2} + \frac{(n)(n+1)}{2}x^{n-1} + \frac{(n+1)(n+2)}{2}x^n + \dots \right)$$

So taking only terms involving x^n we have term

$$\frac{(n+1)(n+2)}{2}x^n + 2\frac{(n)(n+1)}{2}x^n + \frac{(n-1)(n)}{2}x^n \\ = \left((n+1)(n+2) + 2(n)(n+1) + (n-1)(n) \right) \frac{x^n}{2} \\ = \left(n^2 + n + 2n + 2 + 2n^2 + 2n + n^2 - n \right) \frac{x^n}{2} \\ = \left(4n^2 + 4n + 2 \right) \frac{x^n}{2} = 2(2n^2 + 2n + 1) \frac{x^n}{2} \\ = (2n^2 + 2n + 1)x^n$$

Thus the coefficient of term involving x^n is $(2n^2 + 2n + 1)$.

(v) Since we know that

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

Therefore

$$\left(1 - x + x^2 - x^3 + \dots \right)^2 = \left((1+x)^{-1} \right)^2 = (1+x)^{-2} \\ = 1 + (-2)(x) + \frac{(-2)(-2-1)}{2!}(x)^2 + \frac{(-2)(-2-1)(-2-2)}{3!}(x)^3 + \dots \\ = 1 - 2x + \frac{(-2)(-3)}{2}(x)^2 + \frac{(-2)(-3)(-4)}{3 \cdot 2}(x)^3 + \dots \\ = 1 - 2x + 3x^2 - 4x^3 + \dots \\ = 1 + (-1)2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \dots$$

Following in this way we can write

$$= 1 + (-1)2x + (-1)^2 3x^2 + (-1)^3 4x^3 + \dots + (-1)^n (n+1)x^n + \dots$$

So the term involving $x^n = (-1)^n (n+1)x^n$

And hence coefficient of term involving x^n is $(-1)^n (n+1)$

Question # 4

If x so small that its square and higher powers can be neglected, then show that

(i) $\frac{1-x}{\sqrt{1+x}} \approx 1 - \frac{3}{2}x$

(ii) $\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx 1 + \frac{3}{2}x$

(iii) $\frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}}}{4+5x} \approx \frac{1}{4} - \frac{17}{384}x$

(iv) $\frac{\sqrt{4+x}}{(1-x)^3} \approx 2 + \frac{25}{4}x$

$$(v) \frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{1}{4}}}{(8+5x)^{\frac{1}{3}}} \approx \left(1 - \frac{5x}{6}\right)$$

$$(vi) \frac{(1-x)^{\frac{1}{2}}(9-4x)^{\frac{1}{2}}}{(8+3x)^{\frac{1}{3}}} \approx \frac{3}{2} - \frac{61}{48}x$$

$$(vii) \frac{\sqrt{4-x} + (8-x)^{\frac{1}{3}}}{(8-x)^{\frac{1}{3}}} \approx 2 - \frac{1}{12}x$$

Solution

(i)

$$\begin{aligned} \text{L.H.S} &= \frac{1-x}{\sqrt{1+x}} = \frac{1-x}{(1+x)^{\frac{1}{2}}} = (1-x)(1+x)^{-\frac{1}{2}} \\ &= (1-x) \left(1 + \left(-\frac{1}{2}\right)(x) + \text{squares and higher power of } x \right) \\ &= 1-x - \frac{1}{2}x + \text{squares and higher power of } x \\ &\approx 1 - \frac{3}{2}x = \text{R.H.S} \quad \text{Proved} \end{aligned}$$

$$(ii) \quad \text{Since} \quad \frac{\sqrt{1+2x}}{\sqrt{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$\begin{aligned} \text{Now } (1+2x)^{\frac{1}{2}} &= 1 + \left(\frac{1}{2}\right)(2x) + \text{squares and higher power of } x \\ &\approx 1+x \end{aligned}$$

$$\begin{aligned} \text{Now } (1-x)^{-\frac{1}{2}} &= 1 + \left(-\frac{1}{2}\right)(-x) + \text{squares and higher power of } x \\ &\approx 1 + \frac{1}{2}x \end{aligned}$$

$$\frac{\sqrt{1+2x}}{\sqrt{1-x}} \approx (1+x) \left(1 + \frac{1}{2}x \right)$$

$$= 1+x + \frac{1}{2}x$$

ignoring term involving x^2 .

$$= 1 + \frac{3}{2}x \quad \text{Proved.}$$

$$(iii) \quad \frac{(9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}}}{4+5x} = \left((9+7x)^{\frac{1}{2}} - (16+3x)^{\frac{1}{4}} \right) (4+5x)^{-1}$$

$$\text{Now } (9+7x)^{\frac{1}{2}} = 9^{\frac{1}{2}} \left(1 + \frac{7x}{9} \right)^{\frac{1}{2}}$$

$$= (3^2)^{\frac{1}{2}} \left(1 + \left(\frac{1}{2}\right) \left(\frac{7x}{9}\right) + \text{squares and higher of } x \right)$$

$$\approx 3\left(1 + \frac{7x}{18}\right) = 3 + 3\left(\frac{7x}{18}\right) = 3 + \frac{7x}{6}$$

$$\begin{aligned} (16 + 3x)^{\frac{1}{4}} &= (16)^{\frac{1}{4}}\left(1 + \frac{3x}{16}\right)^{\frac{1}{4}} \\ &= (2^4)^{\frac{1}{4}}\left(1 + \left(\frac{1}{4}\right)\left(\frac{3x}{16}\right) + \text{square and higher power of } x\right) \\ &\approx (2)\left(1 + \frac{3x}{64}\right) = 2 + 2\left(\frac{3x}{64}\right) = 2 + \frac{3x}{32} \end{aligned}$$

$$\begin{aligned} (4 + 5x)^{-1} &= 4^{-1}\left(1 + \frac{5}{4}x\right)^{-1} \\ &= \frac{1}{4}\left(1 + (-1)\left(\frac{5}{4}x\right) + \text{squares and higher power of } x\right) \\ &\approx \frac{1}{4}\left(1 - \frac{5}{4}x\right) = \frac{1}{4} - \frac{5}{16}x \end{aligned}$$

$$\begin{aligned} \text{So } \frac{(9 + 7x)^{\frac{1}{2}} - (16 + 3x)^{\frac{1}{4}}}{4 + 5x} &\approx \left[\left(3 + \frac{7x}{6}\right) - \left(2 + \frac{3x}{32}\right)\right]\left(\frac{1}{4} - \frac{5}{16}x\right) \\ &= \left[3 + \frac{7x}{6} - 2 - \frac{3x}{32}\right]\left(\frac{1}{4} - \frac{5}{16}x\right) = \left(1 + \frac{103}{96}x\right)\left(\frac{1}{4} - \frac{5}{16}x\right) \\ &= \frac{1}{4} + \frac{103}{384}x - \frac{5}{16}x = \frac{1}{4} - \frac{17}{384}x \quad \text{Proved} \end{aligned}$$

(iv) *Do yourself*

$$(v) \quad \frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} = (1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}(8+5x)^{-\frac{1}{3}}$$

$$\begin{aligned} \text{Now } (1+x)^{\frac{1}{2}} &= 1 + \left(\frac{1}{2}\right)(x) + \text{square and higher power of } x \\ &\approx 1 + \frac{1}{2}x \end{aligned}$$

$$\begin{aligned} (4-3x)^{\frac{3}{2}} &= 4^{\frac{3}{2}}\left(1 - \frac{3}{4}x\right)^{\frac{3}{2}} \\ &= (2^2)^{\frac{3}{2}}\left(1 + \left(\frac{3}{2}\right)\left(-\frac{3}{4}x\right) + \text{square and higher power of } x\right) \\ &\approx (2)^3\left(1 - \frac{9}{8}x\right) = 8\left(1 - \frac{9}{8}x\right) \end{aligned}$$

$$(8+5x)^{-\frac{1}{3}} = (8)^{-\frac{1}{3}}\left(1 + \frac{5}{8}x\right)^{-\frac{1}{3}}$$

$$= (2^3)^{-\frac{1}{3}} \left(1 + \left(-\frac{1}{3} \right) \left(\frac{5}{8} x \right) + \text{square and higher power of } x \right)$$

$$\approx (2)^{-1} \left(1 - \frac{5}{24} x \right) = \frac{1}{2} \left(1 - \frac{5}{24} x \right)$$

So
$$\frac{(1+x)^{\frac{1}{2}}(4-3x)^{\frac{3}{2}}}{(8+5x)^{\frac{1}{3}}} \approx \left(1 + \frac{1}{2} x \right) 8 \left(1 - \frac{9}{8} x \right) \frac{1}{2} \left(1 - \frac{5}{24} x \right)$$

$$= \frac{8}{2} \left(1 + \frac{1}{2} x \right) \left(1 - \frac{9}{8} x - \frac{5}{24} x \right)$$

$$= 4 \left(1 + \frac{1}{2} x \right) \left(1 - \frac{4}{3} x \right) = 4 \left(1 + \frac{1}{2} x - \frac{4}{3} x \right) = 4 \left(1 - \frac{5}{6} x \right) \quad \text{Proved}$$

(vi) *Do yourself as above*

(vii) *Same as Question #4 (iii)*

Question # 5

If x is so small that its cube and higher power can be neglected, then show that

(i) $\sqrt{1-x-2x^2} = 1 - \frac{1}{2}x - \frac{9}{8}x^2$ (ii) $\sqrt{\frac{1+x}{1-x}} = 1 + x + \frac{1}{2}x^2$

Solution

(i) $\sqrt{1-x-2x^2} = (1 - (x+2x^2))^{\frac{1}{2}}$

$$= 1 + \left(\frac{1}{2} \right) (- (x+2x^2)) + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} (- (x+2x^2))^2 + \text{cube \& higher power of } x.$$

$$\approx 1 - \left(\frac{1}{2} \right) (x+2x^2) + \frac{\frac{1}{2} \left(-\frac{1}{2} \right)}{2} (x+2x^2)^2$$

$$\approx 1 - \frac{1}{2}x - \frac{1}{2}(2x^2) - \frac{1}{8}x^2 = 1 - \frac{1}{2}x - x^2 - \frac{1}{8}x^2$$

$$= 1 - \frac{1}{2}x - \frac{9}{8}x^2 \quad \text{Proved}$$

(ii)

$$\sqrt{\frac{1+x}{1-x}} = \frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

Now

$$(1+x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2} \right) x + \frac{\frac{1}{2} \left(\frac{1}{2} - 1 \right)}{2!} x^2 + \text{cube \& higher power of } x.$$

$$\approx 1 + \frac{1}{2}x + \frac{\frac{1}{2} \left(-\frac{1}{2} \right)}{2} x^2 = 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2 + \text{cube \& higher power of } x.$$

$$\approx 1 + \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2 = 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

So

$$\sqrt{\frac{1+x}{1-x}} = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right) \left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{3}{8}x^2 = 1 + x + \frac{1}{2}x^2 \quad \text{Proved}$$

Question # 6

If x is very nearly equal 1, then prove that $px^p - qx^q = (p - q)x^{p+q}$

Solution

Since x is nearly equal to 1 so suppose $x = 1 + h$,
 where h is so small that its square and higher powers be neglected

$$\begin{aligned} \text{L.H.S} &= px^p - qx^q \\ &= p(1+h)^p - q(1+h)^q \\ &= p(1 + ph + \text{square \& higher power of } x) \\ &\quad - q(1 + qh + \text{square \& higher power of } h) \\ &\approx p(1 + ph) - q(1 + qh) \\ &= p + p^2h - q - q^2h \dots\dots\dots (i) \end{aligned}$$

$$\begin{aligned} \text{Now R.H.S} &= (p - q)x^{p+q} \\ &= (p - q)(1+h)^{p+q} \\ &= (p - q)(1 + (p + q)h + \text{square \& higher power of } h) \\ &\approx (p - q)(1 + (p + q)h) = (p - q)(1 + ph + qh) \\ &= p + p^2h + pqh - q - pqh - q^2h \\ &= p + p^2h - q - q^2h \dots\dots\dots (ii) \end{aligned}$$

From (i) and (ii)
 L.H.S \approx R.H.S Proved

Question # 7

If $p - q$ is small when compared with p or q , show that

$$\frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q} = \left(\frac{p+q}{2q}\right)^{\frac{1}{n}}$$

Solution Since $p - q$ is small when compare

Therefore let $p - q = h \Rightarrow p = q + h$

$$\text{L.H.S} = \frac{(2n+1)p + (2n-1)q}{(2n-1)p + (2n+1)q} = \frac{(2n+1)(q+h) + (2n-1)q}{(2n-1)(q+h) + (2n+1)q}$$

$$\begin{aligned}
 &= \frac{2nq + q + 2nh + h + 2nq - q}{2nq - q + 2nh - h + 2nq + q} = \frac{4nq + 2nh + h}{4nq + 2nh - h} \\
 &= \frac{4nq + 2nh + h}{4nq \left(1 + \frac{2nh - h}{4nq}\right)} = \frac{4nq + 2nh + h}{4nq} \left(1 + \frac{2nh - h}{4nq}\right)^{-1} \\
 &= \frac{4nq + 2nh + h}{4nq} \left(1 + (-1) \left(\frac{2nh - h}{4nq}\right) + \text{square \& higher power of } x^2\right) \\
 &= \frac{4nq + 2nh + h}{4nq} \left(1 - \frac{2nh - h}{4nq}\right) = \frac{4nq + 2nh + h}{4nq} \left(\frac{4nq - 2nh + h}{4nq}\right) \\
 &\approx \frac{16n^2q^2 + 8n^2hq + 4nhq - 8n^2hq + 4nhq}{16n^2q^2} \quad \text{ignoring squares of } h \\
 &= \frac{16n^2q^2 + 8nhq}{16n^2q^2} = \frac{16n^2q^2}{16n^2q^2} + \frac{8nhq}{16n^2q^2} \\
 &= 1 + \frac{h}{2nq} \dots\dots\dots (i)
 \end{aligned}$$

Now R.H.S = $\left(\frac{p+q}{2q}\right)^{\frac{1}{n}} = \left(\frac{q+h+q}{2q}\right)^{\frac{1}{n}}$

$$\begin{aligned}
 &= \left(\frac{2q+h}{2q}\right)^{\frac{1}{n}} = \left(\frac{2q}{2q} + \frac{h}{2q}\right)^{\frac{1}{n}} = \left(1 + \frac{h}{2q}\right)^{\frac{1}{n}} \\
 &= 1 + \left(\frac{1}{n}\right) \left(\frac{h}{2q}\right) + \text{square \& higher power of } h. \\
 &\approx 1 + \frac{h}{2nq} \dots\dots\dots (ii)
 \end{aligned}$$

Form (i) and (ii)
 L.H.S \approx R.H.S Proved

Question # 8

Show that $\left(\frac{n}{2(n+N)}\right)^{\frac{1}{2}} \approx \frac{8n}{9n-N} - \frac{n+N}{4n}$ where n and N are nearly equal.

Solution Since n and N are nearly equal therefore consider $N = n + h$, where h is so small that its squares and higher power be neglected.

$$\begin{aligned}
 \text{L.H.S} &= \left(\frac{n}{2(n+N)}\right)^{\frac{1}{2}} = \left(\frac{n}{2(n+n+h)}\right)^{\frac{1}{2}} \\
 &= \left(\frac{n}{2(2n+h)}\right)^{\frac{1}{2}} = \left(\frac{2(2n+h)}{n}\right)^{-\frac{1}{2}} = \left(\frac{4n+2h}{n}\right)^{-\frac{1}{2}} = \left(4 + \frac{2h}{n}\right)^{-\frac{1}{2}}
 \end{aligned}$$

$$\begin{aligned}
 &= (4)^{-\frac{1}{2}} \left(1 + \frac{2h}{4n}\right)^{-\frac{1}{2}} = (2^2)^{-\frac{1}{2}} \left(1 + \frac{h}{2n}\right)^{-\frac{1}{2}} \\
 &= (2)^{-1} \left(1 + \left(-\frac{1}{2}\right) \frac{h}{2n} + \text{square \& higher power of } h\right) \\
 &= \frac{1}{2} \left(1 - \frac{h}{4n}\right) = \frac{1}{2} - \frac{h}{8n} \dots\dots\dots (i)
 \end{aligned}$$

Now R.H.S = $\frac{8n}{9n-N} - \frac{n+N}{4n}$

$$\begin{aligned}
 &= \frac{8n}{9n-(n+h)} - \frac{n+(n+h)}{4n} = \frac{8n}{9n-n-h} - \frac{n+n+h}{4n} \\
 &= \frac{8n}{8n-h} - \frac{2n+h}{4n} = \frac{8n}{8n\left(1-\frac{h}{8n}\right)} - \frac{2n+h}{4n} = \left(1-\frac{h}{8n}\right)^{-1} - \frac{2n+h}{4n} \\
 &= \left(1 + (-1)\left(-\frac{h}{8n}\right) + \text{square \& higher power of } h\right) - \left(\frac{2n}{4n} + \frac{h}{4n}\right) \\
 &= \left(1 + \frac{h}{8n}\right) - \left(\frac{1}{2} + \frac{h}{4n}\right) = 1 + \frac{h}{8n} - \frac{1}{2} - \frac{h}{4n} \\
 &= \frac{1}{2} - \frac{h}{8n} \dots\dots\dots (ii)
 \end{aligned}$$

From (i) and (ii)

L.H.S = R.H.S Proved

Question # 9

Identify the following series as binomial expansion and find the sum in each case.

(i) $1 - \frac{1}{2} \left(\frac{1}{4}\right) + \frac{1 \cdot 3}{2! \cdot 4} \left(\frac{1}{4}\right)^2 - \frac{1 \cdot 3 \cdot 5}{3! \cdot 8} \left(\frac{1}{4}\right)^3 + \dots$

(ii) $1 - \frac{1}{2} \left(\frac{1}{2}\right) + \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{1}{2}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{2}\right)^3 + \dots$

(iii) $1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \dots$

(iv) $1 - \frac{1}{2} \left(\frac{1}{3}\right) + \frac{1 \cdot 3}{2 \cdot 4} \left(\frac{1}{3}\right)^2 - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \left(\frac{1}{3}\right)^3 + \dots$

Solution

(i) $1 - \frac{1}{2} \left(\frac{1}{4}\right) + \frac{1 \cdot 3}{2! \cdot 4} \left(\frac{1}{4}\right)^2 - \frac{1 \cdot 3 \cdot 5}{3! \cdot 8} \left(\frac{1}{4}\right)^3 + \dots$

Suppose the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

This implies $nx = -\frac{1}{2} \left(\frac{1}{4}\right) \dots\dots\dots (i)$

$$\frac{n(n-1)}{2!}x^2 = \frac{1 \cdot 3}{2! \cdot 4} \left(\frac{1}{4}\right)^2 \dots\dots\dots \text{(ii)}$$

From (i) $nx = -\frac{1}{8} \Rightarrow x = -\frac{1}{8n} \dots\dots\dots \text{(iii)}$

Putting value of x in (ii)

$$\begin{aligned} \frac{n(n-1)}{2!} \left(-\frac{1}{8n}\right)^2 &= \frac{1 \cdot 3}{2! \cdot 4} \left(\frac{1}{4}\right)^2 \\ \Rightarrow \frac{n(n-1)}{2} \left(\frac{1}{64n^2}\right) &= \frac{3}{2 \cdot 4} \left(\frac{1}{16}\right) \\ \Rightarrow \frac{(n-1)}{128n} = \frac{3}{128} &\Rightarrow (n-1) = \frac{3}{128} \cdot 128n \Rightarrow n-1 = 3n \\ \Rightarrow n-3n = 1 &\Rightarrow -2n = 1 \Rightarrow \boxed{n = -\frac{1}{2}} \end{aligned}$$

Putting value of n in equation (iii)

$$x = -\frac{1}{8\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x = \frac{1}{4}}$$

So

$$(1+x)^n = \left(1 + \frac{1}{4}\right)^{-\frac{1}{2}} = \left(\frac{5}{4}\right)^{-\frac{1}{2}} = \left(\frac{4}{5}\right)^{\frac{1}{2}} = \sqrt{\frac{4}{5}}$$

(ii) *Do yourself as above*

(iii) $1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \dots$

Suppose the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

This implies $nx = \frac{3}{4} \dots\dots\dots \text{(i)}$

$$\frac{n(n-1)}{2!}x^2 = \frac{3 \cdot 5}{4 \cdot 8} \dots\dots\dots \text{(ii)}$$

From (i) $nx = \frac{3}{4} \Rightarrow x = \frac{3}{4n} \dots\dots\dots \text{(iii)}$

Putting value of x in (ii)

$$\begin{aligned} \frac{n(n-1)}{2!} \left(\frac{3}{4n}\right)^2 &= \frac{3 \cdot 5}{4 \cdot 8} \Rightarrow \frac{n(n-1)}{2} \left(\frac{9}{16n^2}\right) = \frac{15}{32} \\ \Rightarrow \frac{9(n-1)}{32n} = \frac{15}{32} &\Rightarrow 9(n-1) = \frac{15}{32} \cdot 32n \Rightarrow 9n-9 = 15n \end{aligned}$$

$$\Rightarrow 9n - 15n = 9 \Rightarrow -6n = 9 \Rightarrow n = -\frac{9}{6} \Rightarrow \boxed{n = -\frac{3}{2}}$$

Putting value of n in equation (iii)

$$x = -\frac{3}{4\left(-\frac{3}{2}\right)} \Rightarrow \boxed{x = -\frac{1}{2}}$$

$$\text{So } (1+x)^n = \left(1 - \frac{1}{2}\right)^{-\frac{3}{2}} = \left(\frac{1}{2}\right)^{-\frac{3}{2}} = (2)^{\frac{3}{2}} = (\sqrt{2})^3 = 2\sqrt{2} \text{ Answer}$$

(iv) *Do yourself as above*

Question # 10

Use binomial theorem to show that $1 + \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \dots = \sqrt{2}$

Solution $1 + \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \dots$

Suppose the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots$$

This implies

$$nx = \frac{1}{4} \dots \dots \dots \text{(i)}$$

$$\frac{n(n-1)}{2!}x^2 = \frac{1 \cdot 3}{4 \cdot 8} \dots \dots \dots \text{(ii)}$$

From (i) $nx = \frac{1}{4} \Rightarrow x = \frac{1}{4n} \dots \dots \dots \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!} \left(\frac{1}{4n}\right)^2 = \frac{1 \cdot 3}{4 \cdot 8} \Rightarrow \frac{n(n-1)}{2} \left(\frac{1}{16n^2}\right) = \frac{3}{32}$$

$$\Rightarrow \frac{(n-1)}{32n} = \frac{3}{32} \Rightarrow (n-1) = \frac{3}{32} \cdot 32n \Rightarrow n-1 = 3n$$

$$\Rightarrow n - 3n = 1 \Rightarrow -2n = 1 \Rightarrow \boxed{n = -\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x = \frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x = -\frac{1}{2}}$$

$$\text{So } (1+x)^n = \left(1 - \frac{1}{2}\right)^{-\frac{1}{2}} = \left(\frac{1}{2}\right)^{-\frac{1}{2}} = (2)^{\frac{1}{2}} = \sqrt{2}$$

Hence $1 + \frac{1}{4} + \frac{1 \cdot 3}{4 \cdot 8} + \frac{1 \cdot 3 \cdot 5}{4 \cdot 8 \cdot 12} + \dots = \sqrt{2}$ Proved

Question # 11

If $y = \frac{1}{3} + \frac{1 \cdot 3}{2!} \left(\frac{1}{3}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{1}{3}\right)^3 + \dots$, then prove that $y^2 + 2y - 2 = 0$.

Solution $y = \frac{1}{3} + \frac{1 \cdot 3}{2!} \left(\frac{1}{3}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{1}{3}\right)^3 + \dots$

Adding 1 on both sides

$$1 + y = 1 + \frac{1}{3} + \frac{1 \cdot 3}{2!} \left(\frac{1}{3}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{1}{3}\right)^3 + \dots$$

Let the given series be identical with

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

This implies

$$nx = \frac{1}{3} \dots\dots\dots (i)$$

$$\frac{n(n-1)}{2!} x^2 = \frac{1 \cdot 3}{2!} \left(\frac{1}{3}\right)^2 \dots\dots\dots (ii)$$

From (i) $nx = \frac{1}{3} \Rightarrow x = \frac{1}{3n} \dots\dots\dots (iii)$

Putting value of x in (ii)

$$\begin{aligned} \frac{n(n-1)}{2!} \left(\frac{1}{3n}\right)^2 &= \frac{1 \cdot 3}{2!} \left(\frac{1}{3}\right)^2 \\ \Rightarrow \frac{n(n-1)}{2} \left(\frac{1}{9n^2}\right) &= \frac{3}{2} \cdot \frac{1}{9} \\ \Rightarrow \frac{n-1}{18n} = \frac{1}{6} &\Rightarrow n-1 = \frac{1}{6} \cdot 18n \\ \Rightarrow n-1 = 3n &\Rightarrow n-3n = 1 \\ \Rightarrow -2n = 1 &\Rightarrow \boxed{n = -\frac{1}{2}} \end{aligned}$$

Putting value of n in equation (iii)

$$x = \frac{1}{3\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x = -\frac{2}{3}}$$

So $(1 + x)^n = \left(1 - \frac{2}{3}\right)^{-\frac{1}{2}} = \left(\frac{1}{3}\right)^{-\frac{1}{2}}$
 $= (3)^{\frac{1}{2}} = \sqrt{3}$

This implies

$$1 + y = \sqrt{3}$$

On squaring both sides

$$(1+y)^2 = (\sqrt{3})^2$$

$$\Rightarrow 1+2y+y^2=3 \Rightarrow 1+2y+y^2-3=0$$

$$\Rightarrow y^2+2y-2=0 \quad \text{Proved.}$$

Question # 12

If $2y = \frac{1}{2^2} + \frac{1 \cdot 3}{2!} \cdot \frac{1}{2^4} + \frac{1 \cdot 3 \cdot 5}{3!} \cdot \frac{1}{2^6} + \dots$, then prove that $4y^2 + 4y - 1 = 0$.

Solution $2y = \frac{1}{2^2} + \frac{1 \cdot 3}{2!} \cdot \frac{1}{2^4} + \frac{1 \cdot 3 \cdot 5}{3!} \cdot \frac{1}{2^6} + \dots$

Adding 1 on both sides

$$1+2y = 1 + \frac{1}{2^2} + \frac{1 \cdot 3}{2!} \cdot \frac{1}{2^4} + \frac{1 \cdot 3 \cdot 5}{3!} \cdot \frac{1}{2^6} + \dots$$

Let the given series be identical with

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

This implies

$$nx = \frac{1}{2^2} \dots \dots \dots \text{(i)}$$

$$\frac{n(n-1)}{2!} x^2 = \frac{1 \cdot 3}{2!} \cdot \frac{1}{2^4} \dots \dots \dots \text{(ii)}$$

From (i) $nx = \frac{1}{4} \Rightarrow x = \frac{1}{4n} \dots \dots \dots \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!} \left(\frac{1}{4n}\right)^2 = \frac{1 \cdot 3}{2!} \cdot \frac{1}{2^4}$$

$$\Rightarrow \frac{n(n-1)}{2} \left(\frac{1}{16n^2}\right) = \frac{3}{2} \cdot \frac{1}{16}$$

$$\Rightarrow \frac{n-1}{n} = 3 \Rightarrow n-1 = 3n$$

$$\Rightarrow n-3n=1 \Rightarrow -2n=1 \Rightarrow \boxed{n = -\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x = \frac{1}{4\left(-\frac{1}{2}\right)} \Rightarrow \boxed{x = -\frac{1}{2}}$$

So

$$(1+x)^n = \left(1 - \frac{1}{2}\right)^{-\frac{1}{2}}$$

$$= \left(\frac{1}{2}\right)^{-\frac{1}{2}} = (2)^{\frac{1}{2}} = \sqrt{2}$$

This implies

$$1 + 2y = \sqrt{2}$$

On squaring both sides

$$(1 + 2y)^2 = (\sqrt{2})^2$$

$$\Rightarrow 1 + 4y + 4y^2 = 4 \quad \Rightarrow 1 + 4y + 4y^2 - 4 = 0$$

$$\Rightarrow 4y^2 + 4y - 3 = 0 \quad \text{Proved}$$

Question # 13

If $y = \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$, then prove that $y^2 + 2y - 4 = 0$.

Solution $y = \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$

Adding 1 on both sides

$$1 + y = 1 + \frac{2}{5} + \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 + \frac{1 \cdot 3 \cdot 5}{3!} \left(\frac{2}{5}\right)^3 + \dots$$

Let the given series be identical with

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots$$

This implies

$$nx = \frac{2}{5} \dots \dots \dots \text{(i)}$$

$$\frac{n(n-1)}{2!} x^2 = \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2 \dots \dots \dots \text{(ii)}$$

From (i) $nx = \frac{2}{5} \Rightarrow x = \frac{2}{5n} \dots \dots \dots \text{(iii)}$

Putting value of x in (ii)

$$\frac{n(n-1)}{2!} \left(\frac{2}{5n}\right)^2 = \frac{1 \cdot 3}{2!} \left(\frac{2}{5}\right)^2$$

$$\Rightarrow \frac{n(n-1)}{2} \left(\frac{4}{25n^2}\right) = \frac{3}{2} \left(\frac{4}{25}\right)$$

$$\Rightarrow \frac{n-1}{n} = 3 \quad \Rightarrow n-1 = 3n \quad \Rightarrow n-3n = 1$$

$$\Rightarrow -2n = 1 \quad \Rightarrow \boxed{n = -\frac{1}{2}}$$

Putting value of n in equation (iii)

$$x = \frac{2}{5 \left(-\frac{1}{2}\right)} \Rightarrow \boxed{x = -\frac{4}{5}}$$

$$\begin{aligned}\text{So } (1+x)^n &= \left(1 - \frac{4}{5}\right)^{-\frac{1}{2}} = \left(\frac{1}{5}\right)^{-\frac{1}{2}} \\ &= (5)^{\frac{1}{2}} = \sqrt{5}\end{aligned}$$

This implies

$$1+y = \sqrt{5}$$

On squaring both sides

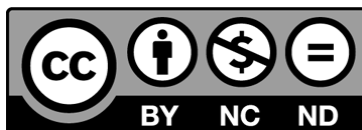
$$\begin{aligned}(1+y)^2 &= (\sqrt{5})^2 \\ \Rightarrow 1+2y+y^2 &= 5 \quad \Rightarrow 1+2y+y^2-5=0 \\ \Rightarrow y^2+2y-4 &= 0 \quad \text{Proved.}\end{aligned}$$

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Error Analyst		
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Book: **Exercise 8.3 (Page 283)**
Text Book of Algebra and Trigonometry Class XI
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