Exercise 4.9

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Case II: Both the Equations are quadratic in two Variables

The equations in this case are classified as

(i) Both the equations contain only x^2 and y^2 terms.

(ii) One of the equations is homogeneous in x and y.

(iii) Both the equations are non-homogeneous.

The methods of solving these type of equations are explained through the following examples.

Example 1: Solve the equations

$$\boldsymbol{x^2 + y^2 = 25; 2x^2 + 3y^2 = 66}$$

Solution: $x^2 + y^2 = 25 \quad \longrightarrow (1)$

$2x^2 + 3y^2 = 66 \quad \longrightarrow (2)$

From equations (1) and (2) we have

$x^2 = 25 - y^2 \quad \longrightarrow (3)$

And $2x^2 = 66 - 3y^2$

$x^2 = \frac{66-3y^2}{2} \quad \longrightarrow (4)$

Now comparing equations (3) and (4),

$$25 - y^2 = \frac{66 - 3y^2}{2}$$

$50 - 2y^2 = 66 - 3y^2$

$3y^2 - 2y^2 = 66 - 50$

$y^2 = 16$

$y = \pm 4$ put in (3), we have

$x^2 = 25 - 16$

$x^2 = 9$

$x = \pm 3$

Hence solution set is $\{(\pm\boldsymbol{3}, \pm\boldsymbol{4})\}$

Example 2: **Solve the equations**

$$\boldsymbol{x^2 - 3xy + 2y^2 = 0;\ 2x^2 - 3x + y^2 = 24}$$

Solution:

$x^2 - 3xy + 2y^2 = 0 \quad \longrightarrow (1)$

$2x^2 - 3x + y^2 = 24 \quad \longrightarrow (2)$

Equation (1) can be written as

$x^2 - 2xy - xy + 2y^2 = 0$

$x(x - 2y) - y(x - 2y) = 0$

$(x - 2y)(x - y) = 0$

$x - 2y = 0 \ \ or \ \ x - y = 0$

$x = 2y \longrightarrow (3) \quad or \quad x = y \longrightarrow (4)$

Putting value from equation (3) in equation (2)

$2(2y)^2 - 3(2y) + y^2 = 24$

$8y^2 - 6y + y^2 = 24$

$9y^2 - 6y - 24 = 0$

Dividing by 3

$3y^2 - 2y - 8 = 0$

$3y^2 - 6y + 4y - 8 = 0$

$3y(y - 2) + 4(y - 2) = 0$

$(y - 2)(3y + 4) = 0$

$y - 2 = 0 \ \ or \ \ 3y + 4 = 0$

$y = 2 \ \ or \ \ y = -\frac{4}{3}$

Putting $y = 2$ in equation (3)

$x = 2(2)$

$x = 4$

Putting $y = -\frac{4}{3}$ in equation (3)

$x = 2\left(-\frac{4}{3}\right)$

$x = -\frac{8}{3}$

Now putting value from equation (4) in equation (2)

$2y^2 - 3y + y^2 = 24$

$3y^2 - 3y - 24 = 0$

Dividing by 3

$y^2 - y - 8 = 0$

Using quadratic formula we have

$$y = \frac{1 \pm \sqrt{1 - 4(1)(-8)}}{2}$$

$$y = \frac{1 \pm \sqrt{1 + 32}}{2}$$

$$y = \frac{1 \pm \sqrt{33}}{2}$$

Putting $y = \frac{1+\sqrt{33}}{2}$ in equation (4)

$x = \frac{1 + \sqrt{33}}{2}$

Putting $y = \frac{1-\sqrt{33}}{2}$ in equation (4)

$x = \frac{1 - \sqrt{33}}{2}$

Hence solution set is
$\left\{\left(-\frac{8}{3}, -\frac{4}{3}\right), \left(\frac{1+\sqrt{33}}{2}, \frac{1+\sqrt{33}}{2}\right), \left(\frac{1-\sqrt{33}}{2}, \frac{1-\sqrt{33}}{2}\right)\right\}$

Example 3: Solve the equations

$\boldsymbol{x^2 - y^2 = 5;\ 4x^2 - 3xy = 18}$

Solution: $x^2 - y^2 = 5 \quad \longrightarrow (1)$

$4x^2 - 3xy = 18 \quad \longrightarrow (2)$

Multiplying equation (1) by 18 and (2) by 5 then subtracting

$$\begin{array}{rcl} 18x^2 - 18y^2 & & = 90 \\ 20x^2 & - 15xy & = 90 \\ - & + & - \\ \hline -2x^2 - 18y^2 + 15xy = 0 \end{array}$$

$2x^2 - 15xy + 18y^2 = 0$

$2x^2 - 12xy - 3xy + 18y^2 = 0$

$2x(x - 6y) - 3y(x - 6y) = 0$

$(x - 6y)(2x - 3y) = 0$

$x - 6y = 0 \ \ or \ \ 2x - 3y = 0$

$x = 6y \longrightarrow (3) \quad or \quad x = \frac{3y}{2} \longrightarrow (4)$

Putting value from equation (3) in equation (1)

$(6y)^2 - y^2 = 5$

$36y^2 - y^2 = 5$

$35y^2 = 5$

$y^2 = \frac{5}{35}$

$y^2 = \frac{1}{7}$

$y = \pm\frac{1}{\sqrt{7}}$

Putting $y = \frac{1}{\sqrt{7}}$ in (3)

$x = \frac{6}{\sqrt{7}}$

Put $y = -\frac{1}{\sqrt{7}}$ in (3)

$x = -\frac{6}{\sqrt{7}}$

Now putting value from equation (4) in equation (1)

$\left(\frac{3y}{2}\right)^2 - y^2 = 5$

$\frac{9y^2}{4} - y^2 = 5$

Multiplying by 4

$9y^2 - 4y^2 = 20$

$5y^2 = 20$

$y^2 = 4$

$y = \pm 2$

Putting $y = 2$ in (4)

$x = \dfrac{3(2)}{2}$

$x = 3$

Putting $y = -2$ in equation (4)

$x = \dfrac{3(-2)}{2}$

Hence solution set is
$\left\{\left(\frac{6}{\sqrt{7}}, \frac{1}{\sqrt{7}}\right), \left(-\frac{6}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\right), (3, 2), (-3, -2)\right\}$

Exercise

Solve the following system of equations.

Q#1: $\mathbf{2x^2 = 6 + 3y^2;\ 3x^2 - 5y^2 = 7.}$

Solution:

$2x^2 = 6 + 3y^2 \quad \longrightarrow (1)$

$3x^2 - 5y^2 = 7 \quad \longrightarrow (2)$

From (1) and (2) we can write:

$x^2 = \frac{6+3y^2}{2} \quad \longrightarrow (3)$

$x^2 = \frac{7+5y^2}{3} \quad \longrightarrow (4)$

Comparing equations (3) and (4), we get

$\frac{7+5y^2}{3} = \frac{6+3y^2}{2}$

$14 + 10y^2 = 18 + 9y^2$

$10y^2 - 9y^2 = 18 - 14$

$y^2 = 4$

$y = \pm 2$

Putting value in (3); $x^2 = \frac{6+12}{2}$

$x^2 = 9$

$x = \pm 3$

Hence solution set is $\{(\pm 3, \pm 2)\}$

Q#2: $\mathbf{8x^2 = y^2;\ x^2 + 2y^2 = 19.}$

Solution:

$8x^2 = y^2 \quad \longrightarrow (1)$

$x^2 + 2y^2 = 19 \quad \longrightarrow (2)$

From (1) and (2) we can write:

$x^2 = \frac{y^2}{8} \quad \longrightarrow (3)$

$x^2 = 19 - 2y^2 \quad \longrightarrow (4)$

Comparing equations (3) and (4), we get

$\frac{y^2}{8} = 19 - 2y^2$

$y^2 = 152 - 16y^2$

$16y^2 + y^2 = 152$

$17y^2 = 152$

$y^2 = \dfrac{152}{17}$

$y = \pm\sqrt{\dfrac{152}{17}}$

$y = \pm 2\sqrt{\dfrac{38}{17}}$

Putting value in (3); $x^2 = \frac{\frac{152}{17}}{8}$

$x^2 = \dfrac{19}{17}$

$x = \pm\sqrt{\dfrac{19}{17}}$

Hence solution set is $\left\{\left(\pm\sqrt{\frac{19}{17}}, \pm 2\sqrt{\frac{38}{17}}\right)\right\}$

Q#3: $\mathbf{2x^2 - 8 = 5y^2;\ x^2 - 13 = -2y^2.}$

Solution:

$$2x^2 - 8 = 5y^2 \quad \longrightarrow (1)$$

$$x^2 - 13 = -2y^2 \quad \longrightarrow (2)$$

From (1) and (2) we can write:

$$x^2 = \frac{8+5y^2}{2} \quad \longrightarrow (3)$$
$$x^2 = 13 - y^2 \quad \longrightarrow (4)$$

Comparing equations (3) and (4), we get

$$\frac{8+5y^2}{2} = 13 - 2y^2$$

$$8 + 5y^2 = 26 - 4y^2$$

$$5y^2 + 4y^2 = 26 - 8$$

$$9y^2 = 18$$

$$y^2 = 2$$
$$y = \pm\sqrt{2}$$

Putting value in (4); $x^2 = 13 - 4$

$$x^2 = 9$$
$$x = \pm 3$$

Hence solution set is $\{(\pm 3, \pm\sqrt{2})\}$

Q#4: $\boldsymbol{x^2 - 5xy + 6y^2 = 0;\ x^2 + y^2 = 45.}$

Solution:

$$x^2 - 5xy + 6y^2 = 0 \quad \longrightarrow (1)$$

$$x^2 + y^2 = 45 \quad \longrightarrow (2)$$

From (1)

$$x^2 - 2xy - 3xy + 6y^2 = 0$$

$$x(x - 2y) - 3y(x - 2y) = 0$$

$$(x - 2y)(x - 3y) = 0$$

$$x - 2y = 0 \text{ or } x - 3y = 0$$

$$x = 2y \longrightarrow (3) \qquad \text{or } x = 3y \longrightarrow (4)$$

Putting value from (3) in (2), we get

$$(2y)^2 + y^2 = 45$$

$$4y^2 + y^2 = 45$$

$$5y^2 = 45$$

$$y^2 = 9$$

$y = \pm 3$ put in (3)

$$x = 2(\pm 3) = \pm 6$$

Putting value from (4) in (2), we get

$$(3y)^2 + y^2 = 45$$

$$9y^2 + y^2 = 45$$

$$10y^2 = 45$$

$$y^2 = \frac{45}{10}$$

$$y^2 = \frac{9}{2}$$

$$y = \pm\frac{3}{\sqrt{2}}$$

Putting $y = \frac{3}{\sqrt{2}}$ in (4)

$$x = 3\left(\frac{3}{\sqrt{2}}\right) = \frac{9}{\sqrt{2}}$$

Putting $y = -\frac{3}{\sqrt{2}}$ in (4)

$$x = 3\left(-\frac{3}{\sqrt{2}}\right) = -\frac{9}{\sqrt{2}}$$

Hence solution set is
$\left\{(\pm 6,\ \pm 3), \left(\frac{9}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right), \left(-\frac{9}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)\right\}$

Q#5: $\boldsymbol{12x^2 - 25xy + 12y^2 = 04x^2 + 7y^2 = 148}$

Solution:

$$12x^2 - 25xy + 12y^2 = 0 \quad \longrightarrow (1)$$

$$4x^2 + 7y^2 = 148 \quad \longrightarrow (2)$$

From (1):

$$12x^2 - 16xy - 9xy + 12y^2 = 0$$

$$4x(3x - 4y) - 3y(3x - 4y) = 0$$

$$(3x - 4y)(4x - 3y) = 0$$

$3x - 4y = 0$ or $4x - 3y = 0$

$x = \frac{4y}{3} \longrightarrow (3)$ or $x = \frac{3y}{4} \longrightarrow (4)$

Putting value from (3) in (2), we get

$$4\left(\frac{4y}{3}\right)^2 + 7y^2 = 148$$

$$4\left(\frac{16y^2}{9}\right) + 7y^2 = 148$$

$$\frac{64y^2 + 63y^2}{9} = 148$$

$$\frac{127y^2}{9} = 148$$

$$y^2 = \frac{148 \times 9}{127}$$

$$y = \pm\frac{6\sqrt{37}}{\sqrt{127}}$$

$$y = \pm 6\sqrt{\frac{37}{127}}$$

Putting $y = 6\sqrt{\frac{37}{127}}$ in (3)

$$x = \frac{4}{3}\left(6\sqrt{\frac{37}{127}}\right) = 8\sqrt{\frac{37}{127}}$$

Putting $y = -6\sqrt{\frac{37}{127}}$ in (3)

$$x = \frac{4}{3}\left(-6\sqrt{\frac{37}{127}}\right) = -8\sqrt{\frac{37}{127}}$$

Putting value from (4) in (2), we get

$$4\left(\frac{3y}{4}\right)^2 + 7y^2 = 148$$

$$4\left(\frac{9y^2}{16}\right) + 7y^2 = 148$$

$$\frac{9y^2 + 28y^2}{4} = 148$$

$$\frac{37y^2}{4} = 148$$

$$y^2 = \frac{148 \times 4}{37}$$

$$y = \pm 4$$

Putting $y = 4$ in (4)

$$x = \frac{3}{4}(4) = 3$$

Putting $y = -4$ in (4)

$$x = \frac{3}{4}(-4) = -3$$

Hence solution set is

$$\left\{(3,4), (-3,-4), \left(8\sqrt{\frac{37}{127}}, 6\sqrt{\frac{37}{127}}\right), \left(-8\sqrt{\frac{37}{127}}, -6\sqrt{\frac{37}{127}}\right)\right\}$$

Q#6: $\mathbf{12x^2 - 11xy + 2y^2 = 0;\ 2x^2 + 7xy = 60}$

Solution: $12x^2 - 11xy + 2y^2 = 0 \longrightarrow (1)$

$2x^2 + 7xy = 60 \longrightarrow (2)$

Equation (1) can be written as

$$12x^2 - 8xy - 3xy + 2y^2 = 0$$

$$4x(3x - 2y) - y(3x - 2y) = 0$$

$$(3x - 2y)(4x - y) = 0$$

$3x - 2y = 0$ *or* $4x - y = 0$

$3x = 2y$ *or* $4x = y$

$x = \frac{2y}{3} \longrightarrow (3)$ *or* $x = \frac{y}{4} \longrightarrow (4)$

Putting value from (3) in (2), we have

$$2\left(\frac{2y}{3}\right)^2 + 7\left(\frac{2y}{3}\right)y = 60$$

$$2\left(\frac{4y^2}{9}\right) + \left(\frac{14y}{3}\right)y = 60$$

$$\frac{8y^2}{9} + \frac{14y^2}{3} = 60$$

$$\frac{8y^2 + 42y^2}{9} = 60$$

$$8y^2 + 42y^2 = 540$$

$$50y^2 = 540$$

$$y^2 = \frac{54}{5}$$

$$y = \pm\sqrt{\frac{54}{5}}$$

$$y = \pm\frac{\sqrt{54}}{\sqrt{5}}$$

$$y = \pm\frac{3\sqrt{6}\sqrt{5}}{\sqrt{5}\sqrt{5}}$$

$y = \pm\frac{3\sqrt{30}}{5}$ put in (3)

$$x = \frac{2}{3}\left(\pm\frac{3\sqrt{30}}{5}\right)$$

$$x = \pm\frac{2\sqrt{30}}{5}$$

Putting value from (4) in (2)

$$2\left(\frac{y}{4}\right)^2 + 7\left(\frac{y}{4}\right)y = 60$$

$$2\left(\frac{y^2}{4}\right) + \left(\frac{7y}{4}\right)y = 60$$

$$\frac{y^2}{2} + \frac{7y^2}{4} = 60$$

$$\frac{2y^2 + 7y^2}{4} = 60$$

$$2y^2 + 7y^2 = 240$$

$$9y^2 = 240$$

$$y^2 = \frac{240}{9}$$

$$y = \pm\sqrt{\frac{240}{9}}$$

$y = \pm\frac{4\sqrt{15}}{3}$ put in (4)

$$x = \frac{\pm\frac{4\sqrt{15}}{3}}{4}$$

$$x = \pm\frac{\sqrt{15}}{3}$$

Hence solution set is

$$\left\{\left(\pm\frac{\sqrt{15}}{3}, \pm\frac{4\sqrt{15}}{3}\right), \left(\pm\frac{2\sqrt{30}}{5}, \pm\frac{3\sqrt{30}}{5}\right)\right\}$$

Q#7: $\boldsymbol{x^2 - y^2 = 16;\ xy = 15}$

Solution:

$$x^2 - y^2 = 16 \quad \longrightarrow (1)$$

$$xy = 15 \quad \longrightarrow (2)$$

Multiplying equation (1) by 15 and equation (2) by 16 and then subtracting

$$15x^2 - 15y^2 \qquad = 240$$
$$16\,xy = 240$$
$$- \qquad -$$

$$15x^2 - 15y^2 - 16xy = 0$$

$$15x^2 - 16xy - 15y^2 = 0$$

$$15x^2 - 25xy + 9xy - 15y^2 = 0$$

$$5x(3x - 5y) + 3y(3x - 5y) = 0$$

$$(3x - 5y)(5x + 3y) = 0$$

$$3x - 5y = 0 \ \ or \ \ 5x + 3y = 0$$

$$3x = 5y \ \ or \ \ 5x = -3y$$

$$x = \frac{5}{3}y \longrightarrow (3) \ \ or \ \ x = -\frac{3}{5}y \longrightarrow (4)$$

Putting value from equation (3) in equation (2), we have

$$\left(\frac{5}{3}y\right)y = 15$$

On multiplying by 3

$$5y^2 = 45$$

$$y^2 = 9$$

$$y = \pm 3$$

Putting $y = \pm 3$ in equation (3)

$$x = \frac{5}{3}(\pm 3)$$

$$x = \pm 5$$

Now putting value from equation (4) in equation (2)

$$\left(-\frac{3}{5}y\right)y = 15$$

On multiplying by 5

$$-3y^2 = 75$$

$$y^2 = -\frac{75}{3}$$

$$y^2 = -25$$

$$y = \pm 5i$$

Putting $y = \pm 5i$ in equation (4)

$$x = -\frac{3}{5}(\pm 5i)$$

$$x = \pm 3i$$

Hence solution set is $\{(\pm 5, \pm 3), (\pm 3i, \pm 5i)\}$

Q#8: $\boldsymbol{x^2 + xy = 9;\ x^2 - y^2 = 2}$

Solution: $x^2 + xy = 9 \quad \longrightarrow (1)$

$x^2 - y^2 = 2 \quad \longrightarrow (2)$

Multiplying equation (1) by 2 and equation (2) by 9 and then subtracting

$$\begin{array}{l} 2x^2 + 2xy \qquad = 18 \\ 9x^2 \qquad - 9y^2 = 18 \\ - \qquad\quad + \qquad - \\ \hline -7x^2 + 2xy + 9y^2 = 0 \end{array}$$

$$-(7x^2 - 2xy - 9y^2) = 0$$

$$7x^2 - 2xy - 9y^2 = 0$$

$$7x^2 - 9xy + 7xy - 9y^2 = 0$$

$$x(7x - 9y) + y(7x - 9y) = 0$$

$$(7x - 9y)(x + y) = 0$$

$$7x - 9y = 0 \ \ or\ \ x + y = 0$$

$$7x = 9y \ \ or\ \ x = -y$$

$$x = \frac{9}{7}y \longrightarrow (3) \ \ or\ \ x = -y \ \longrightarrow (4)$$

Putting value from equation (3) in equation (2), we have

$$\left(\frac{9}{7}y\right)^2 - y^2 = 2$$

$$\frac{81}{49}y^2 - y^2 = 2$$

On multiplying by 49

$$81y^2 - 49y^2 = 98$$

$$32y^2 = 98$$

$$y^2 = \frac{98}{32}$$

$$y^2 = \frac{49}{16}$$

$$y = \pm\frac{7}{4}$$

Putting $y = \pm\frac{7}{4}$ in equation (3)

$$x = \frac{9}{7}\left(\pm\frac{7}{4}\right)$$

$$x = \pm\frac{9}{4}$$

Now putting value from equation (4) in equation (2)

$$(-y\,)^2 - y^2 = 2$$

$$y^2 - y^2 = 2$$

$$0 = 2$$

Which is not possible.

Hence solution set is $\left\{\left(\pm\frac{9}{4}, \pm\frac{7}{4}\right)\right\}$

Q#9: $\boldsymbol{y^2 - 7 = 2xy\ ;\ 2x^2 + 3 = xy}$

Solution: $y^2 - 7 = 2xy \quad \longrightarrow (1)$

$2x^2 + 3 = xy \quad \longrightarrow (2)$

Multiplying equation (1) by 3 and equation (2) by 7 and then adding

$$\begin{array}{l} 3y^2 - 21 \qquad\quad = 6xy \\ \qquad\ \ 21 + 14x^2 = 7xy \\ \hline 3y^2 + 14x^2 = 13xy \end{array}$$

$$14x^2 - 13xy + 3y^2 = 0$$

$$14x^2 - 7xy - 6xy + 3y^2 = 0$$

$$7x(2x - y) - 3y(2x - y) = 0$$

$$(2x - y)(7x - 3y) = 0$$

$$2x - y = 0 \ \ or\ \ 7x - 3y = 0$$

$$2x = y \ \ or\ \ 7x = 3y$$

$$x = \frac{y}{2} \longrightarrow (3) \quad or \quad x = \frac{3y}{7} \longrightarrow (4)$$

Putting value from equation (3) in equation (2), we have

$$2\left(\frac{y}{2}\right)^2 + 3 = \left(\frac{y}{2}\right)y$$

$$\frac{2y^2}{4} + 3 = \frac{y^2}{2}$$

$$\frac{y^2}{2} + 3 = \frac{y^2}{2}$$

On multiplying by 2

$$y^2 + 6 = y^2$$

$$6 = 0$$

$w$hich is not possible

Now putting value from equation (4) in equation (2)

$$2\left(\frac{3y}{7}\right)^2 + 3 = \left(\frac{3y}{7}\right)y$$

$$\frac{18y^2}{49} + 3 = \frac{3y^2}{7}$$

Multiplying by 49

$$18y^2 + 147 = 21y^2$$

$$3y^2 = 147$$

$$y^2 = 49$$

$$y = \pm 7$$

Putting $y = \pm 7$ in (4)

$$x = \frac{3(\pm 7)}{7}$$

$$x = \pm 3$$

Hence solution set is $\{(\pm 3, \pm 7)\}$

Q#10: $\boldsymbol{x^2 + y^2 = 5 \; ; \; xy = 2}$

Solution: $x^2 + y^2 = 5 \quad \longrightarrow (1)$

$xy = 2 \quad \longrightarrow (2)$

Multiplying equation (1) by 2 and equation (2) by 5 and then subtracting

$$\begin{array}{r} 2x^2 + 2y^2 \quad = 10 \\ 5xy = 10 \\ - \quad\quad - \\ \hline 2x^2 + 2y^2 - 5xy = 0 \end{array}$$

$$2x^2 - 5xy + 2y^2 = 0$$

$$2x^2 - 4xy - xy + 2y^2 = 0$$

$$2x(x - 2y) - y(x - 2y) = 0$$

$$(x - 2y)(2x - y) = 0$$

$$x - 2y = 0 \quad or \quad 2x - y = 0$$

$$x = 2y \quad or \quad 2x = y$$

$$x = 2y \longrightarrow (3) \quad or \quad x = \frac{y}{2} \longrightarrow (4)$$

Putting value from equation (3) in equation (2), we have

$$(2y)y = 2$$

$$2y^2 = 2$$

$$y^2 = 1$$

$$y = \pm 1$$

Putting $y = \pm 1$ in (3)

$$x = 2(\pm 1)$$

$$x = \pm 2$$

Now putting value from equation (4) in equation (2)

$$\left(\frac{y}{2}\right)y = 2$$

$$\frac{y^2}{2} = 2$$

$$y^2 = 4$$

$$y = \pm 2$$

Putting $y = \pm 2$ in (4)

$$x = \frac{\pm 2}{2}$$

$$x = \pm 1$$

Hence solution set is $\{(\pm 2, \pm 1), (\pm 1, \pm 2)\}$