Exercise 4.8

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System of Two Equations Involving Two Variables

We have, so far, been solving quadratic equations in one variable. Now we shall be solving the equations in two variables, when at least one of them is quadratic. To determine the values of two variables, we need two equations. Such a pair of equations is called a *system of simultaneous equations*.

Case I: One Linear Equation and one Quadratic Equation

If one of the equations is linear, we can find the value of one variable in term of the other variable from the linear equation. Substituting this value of one variable in the quadratic equation, we can solve it. The procedure is illustrated through the following examples.

Example 1

Solve the system of equations

$$\boldsymbol{x + y = 7 \text{ and } x^2 - xy + y^2 = 13}$$

Solution:

$x + y = 7 \quad \longrightarrow (1)$

$x^2 - xy + y^2 = 13 \quad \longrightarrow (2)$

From equation (1)

$y = 7 - x \quad \longrightarrow (3)$

Putting value from (3) in (2)

$$x^2 - x(7 - x) + (7 - x)^2 = 13$$

$$x^2 - 7x + x^2 + 49 - 14x + x^2 = 13$$

$$3x^2 - 21x + 49 - 13 = 0$$

$$3x^2 - 21x + 36 = 0$$

$$3(x^2 - 7x + 12) = 0$$

$$x^2 - 7x + 12 = 0$$

$$x^2 - 3x - 4x + 12 = 0$$

$$x(x - 3) - 4(x - 3) = 0$$

$$(x - 3)(x - 4) = 0$$

$$x - 3 = 0 \ or \ x - 4 = 0$$

$$x = 3 \ or \ x = 4$$

Putting $x = 3$ in (3)

$$y = 7 - 3 = 4$$

Putting $x = 4$ in (3)

$$y = 7 - 4 = 3$$

Hence solution set is $\{(3, 4), (4, 3)\}$

Example 2: Solve the following equations

$$\boldsymbol{x^2 + y^2 + 4x = 1 \text{ and } x^2 + (y - 1)^2 = 10}$$

Solution:

$x^2 + y^2 + 4x = 1 \quad \longrightarrow (1)$

$x^2 + (y - 1)^2 = 10 \quad \longrightarrow (2)$

From equation (2)

$x^2 + y^2 - 2y + 1 = 10$

$x^2 + y^2 - 2y = 9 \quad \longrightarrow (3)$

Subtracting equations (1) and (3)

$$\begin{array}{rrrrr} x^2 & + y^2 & + 4x & & = 1 \\ x^2 & + y^2 & & - 2y & = 9 \\ - & - & & + & - \\ \hline \end{array}$$

$$4x + 2y = -8$$

$$2x + y = -4$$

$$y = -2x - 4 \quad \longrightarrow (4)$$

Putting value from equation (4) in (1)

$$x^2 + (-2x - 4)^2 + 4x = 1$$

$$x^2 + 4x^2 + 16x + 16 + 4x = 1$$

$$5x^2 + 20x + 15 = 0$$

$$5(x^2 + 4x + 3) = 0$$

$$x^2 + 4x + 3 = 0$$

$$x^2 + 3x + x + 3 = 0$$

$$x(x + 3) + 1(x + 3) = 0$$

$$(x + 3)(x + 1) = 0$$

$$x + 3 = 0 \;\; or \;\; x + 1 = 0$$

$$x = -3 \;\; or \;\; x = -1$$

Putting $x = -3$ in (3)

$$y = -2(-3) - 4 = 2$$

Putting $x = -1$ in (3)

$$y = -2(-1) - 4 = -2$$

Hence solution set is $\{(-3, 2), (-1, -2)\}$

Exercise

Solve the following systems of equations.

Q#1: $\boldsymbol{2x - y = 4;\; 2x^2 - 4xy - y^2 = 6}$

Solution:

$$2x - y = 4 \longrightarrow (1)$$

$$2x^2 - 4xy - y^2 = 6 \longrightarrow (2)$$

From equation (1)

$$y = 2x - 4 \longrightarrow (3)$$

Putting value from (3) in (1), we have

$$2x^2 - 4x(2x - 4) - (2x - 4)^2 = 6$$

$$2x^2 - 8x^2 + 16x - (4x^2 - 16x + 16) = 6$$

$$-6x^2 + 16x - 4x^2 + 16x - 16 - 6 = 0$$

$$-10x^2 + 32x - 22 = 0$$

$$-2(5x^2 - 16x + 11) = 0$$

$$5x^2 - 16x + 11 = 0$$

$$5x^2 - 5x - 11x + 11 = 0$$

$$5x(x - 1) - 11(x - 1) = 0$$

$$(x - 1)(5x - 11) = 0$$

$$x - 1 = 0 \;\; or \;\; 5x - 11 = 0$$

$$x = 1 \;\; or \;\; 5x = 11$$

$$x = 1 \;\; or \;\; x = \frac{11}{5}$$

Putting $x = 1$ in (3)

$$y = 2(1) - 4 = -2$$

Putting $x = 1$ in (3)

$$y = 2\left(\frac{11}{5}\right) - 4 = \frac{22}{5} - 4 = \frac{22 - 20}{5} = \frac{-2}{5}$$

Hence solution set is $\left\{(1, -2), \left(\frac{11}{5}, \frac{-2}{5}\right)\right\}$

Q#2: $x + y = 5;\; x^2 + 2y^2 = 17$

Solution

$$x+y=5 \dashrightarrow (1)$$

$$x^2+2y^2=17 \dashrightarrow (2)$$

From equation (1)

$$y=5-x \dashrightarrow (3)$$

Putting value from (3) in (1), we have

$$x^2+2(5-x)^2=17$$

$$x^2+2(25+x^2-10x)=17$$

$$x^2+50+2x^2-20x-17=0$$

$$3x^2-20x+33=0$$

$$3x^2-11x-9x+33=0$$

$$x(3x-11)-3(3x-11)=0$$

$$(3x-11)(x-3)=0$$

$$3x-11=0 \;\; or \;\; x-3=0$$

$$3x=11 \;\; or \;\; x=3$$

$$x=\frac{11}{3} \;\; or \;\; x=3$$

Putting $x=\frac{11}{3}$ in (3)

$$y=5-\frac{11}{3}=\frac{15-11}{3}=\frac{4}{3}$$

Putting $x=3$ in (3)

$$y=5-3=2$$

Hence solution set is $\left\{\left(\frac{11}{3},\frac{4}{3}\right),(3,2)\right\}$

Q#3: $3x+2y=7;\ 3x^2=25+2y^2$

Solution

$$3x+2y=7 \dashrightarrow (1)$$

$$3x^2=25+2y^2 \dashrightarrow (2)$$

From equation (1)

$$y=\frac{7-3x}{2} \dashrightarrow (3)$$

Putting value from (3) in (1), we have

$$3x^2=25+2\left(\frac{7-3x}{2}\right)^2$$

$$3x^2=25+2\left(\frac{49-42x+9x^2}{4}\right)$$

$$3x^2=25+\left(\frac{49-42x+9x^2}{2}\right)$$

$$6x^2=50+49-42x+9x^2$$

$$6x^2-9x^2+42x-50-49=0$$

$$-3x^2+42x-99=0$$

$$3x^2-42x+99=0$$

$$x^2-14x+33=0$$

$$x^2-11x-3x+33=0$$

$$x(x-11)-3(x-11)=0$$

$$(x-11)(x-3)=0$$

$$x-11=0 \;\; or \;\; x-3=0$$

$$x=11 \;\; or \;\; x=3$$

Putting $x=11$ in (3)

$$y=\frac{7-3(11)}{2}=\frac{7-33}{2}=\frac{-26}{2}=-13$$

Putting $x=3$ in (3)

$$y=\frac{7-3(3)}{2}=\frac{7-9}{2}=\frac{-2}{2}=-1$$

Hence solution set is $\{(11,-13),(3,-1)\}$

Q#4: $x+y=5;\ \frac{2}{x}+\frac{3}{y}=2$

Solution

$$x + y = 5 \longrightarrow (1)$$

$$\frac{2}{x} + \frac{3}{y} = 2 \longrightarrow (2)$$

From equation (1)

$$y = 5 - x \longrightarrow (3)$$

Putting value from (3) in (1), we have

$$\frac{2}{x} + \frac{3}{5-x} = 2$$

On multiplying by $x(5-x)$

$$2(5-x) + 3x = 2x(5-x)$$

$$10 - 2x + 3x = 10x - 2x^2$$

$$2x^2 - 10x + x + 10 = 0$$

$$2x^2 - 9x + 10 = 0$$

$$2x^2 - 4x - 5x + 10 = 0$$

$$2x(x-2) - 5(x-2) = 0$$

$$(x-2)(2x-5) = 0$$

$$x - 2 = 0 \ \ or \ \ 2x - 5 = 0$$

$$x = 2 \ \ or \ \ 2x = 5$$

$$x = 2 \ \ or \ \ x = \frac{5}{2}$$

Putting $x = 2$ in (3)

$$y = 5 - 2 = 3$$

Putting $x = \frac{5}{2}$ in (3)

$$y = 5 - \frac{5}{2} = \frac{10-5}{2} = \frac{5}{2}$$

Hence solution set is $\left\{(2,3), \left(\frac{5}{2}, \frac{5}{2}\right)\right\}$

Q#5: $x + y = a + b; \frac{a}{x} + \frac{b}{y} = 2$

Solution

$$x + y = a + b \longrightarrow (1)$$

$$\frac{a}{x} + \frac{b}{y} = 2 \longrightarrow (2)$$

From equation (1)

$$y = a + b - x \longrightarrow (3)$$

Putting value from (3) in (1), we have

$$\frac{a}{x} + \frac{b}{a+b-x} = 2$$

On multiplying by $x(a+b-x)$

$$a(a+b-x) + bx = 2x(a+b-x)$$

$$a^2 + ab - ax + bx = 2ax + 2bx - 2x^2$$

$$2x^2 - 2ax - 2bx - ax + bx + a^2 + ab = 0$$

$$2x^2 - 3ax - bx + a^2 + ab = 0$$

$$2x^2 - (3a+b)x + a^2 + ab = 0$$

Using quadratic formula

$$x = \frac{(3a+b) \pm \sqrt{(3a+b)^2 - 4(2)(a^2+ab)}}{4}$$

$$x = \frac{(3a+b) \pm \sqrt{9a^2 + b^2 + 6ab - 8a^2 - 8ab}}{4}$$

$$x = \frac{(3a+b) \pm \sqrt{a^2 + b^2 - 2ab}}{4}$$

$$x = \frac{(3a+b) \pm \sqrt{(a-b)^2}}{4}$$

$$x = \frac{(3a+b) \pm (a-b)}{4}$$

$$x = \frac{3a+b+(a-b)}{4}, x = \frac{(3a+b)-(a-b)}{4}$$

$$x = \frac{3a+b+a-b}{4}, x = \frac{3a+b-a+b}{4}$$

$$x = \frac{4a}{4}, x = \frac{2a+2b}{4}$$

$$x = a, x = \frac{a+b}{2}$$

Putting $x = a$ in (3)

$$y = a + b - a$$

$$y = b$$

Putting $x = \frac{a+b}{2}$ in (3)

$$y = a + b - \frac{a+b}{2}$$

$$y = \frac{2a + 2b - a - b}{2}$$

$$y = \frac{a+b}{2}$$

Hence solution set is $\left\{(a,b), \left(\frac{a+b}{2}, \frac{a+b}{2}\right)\right\}$

Q#6: $3x + 4y = 25; \frac{3}{x} + \frac{4}{y} = 2$

Solution

$$3x + 4y = 25 \longrightarrow (1)$$

$$\frac{3}{x} + \frac{4}{y} = 2 \longrightarrow (2)$$

From equation (1)

$$y = \frac{25 - 3x}{4} \longrightarrow (3)$$

Putting value from (3) in (1), we have

$$\frac{3}{x} + \frac{4}{\frac{25-3x}{4}} = 2$$

$$\frac{3}{x} + \frac{16}{25-3x} = 2$$

Multiplying by $x(25 - 3x)$, we have

$$3(25 - 3x) + 16x = 2x(25 - 3x)$$

$$75 - 9x + 16x = 50x - 6x^2$$

$$6x^2 - 50x - 9x + 16x + 75 = 0$$

$$6x^2 - 43x + 75 = 0$$

$$6x^2 - 25x - 18x + 75 = 0$$

$$x(6x - 25) - 3(6x - 25) = 0$$

$$(6x - 25)(x - 3) = 0$$

$$6x - 25 = 0 \ or \ x - 3 = 0$$

$$6x = 25 \ or \ x = 3$$

$$x = \frac{25}{6} \ or \ x = 3$$

Putting $x = \frac{25}{6}$ in (3)

$$y = \frac{25 - 3\left(\frac{25}{6}\right)}{4} = \frac{25 - \frac{25}{2}}{4}$$

$$= \frac{\frac{50-25}{2}}{4} = \frac{25}{8}$$

Putting $x = 3$ in (3)

$$y = \frac{25 - 3(3)}{4} = \frac{25 - 9}{4} = 4$$

Hence solution set is $\left\{\left(\frac{25}{6}, \frac{25}{8}\right), (3, 4)\right\}$

Q#7: $(x-3)^2 + y^2 = 5; 2x = y + 6$

Solution:

$(x-3)^2 + y^2 = 5 \qquad \longrightarrow (1)$

$2x = y + 6 \qquad \longrightarrow (2)$

From equation (2)

$y = 2x - 6 \qquad \longrightarrow (3)$

Putting value from equation (3) in (1)

$(x-3)^2 + (2x-6\,)^2 = 5$

$$x^2 - 6x + 9 + 4x^2 - 24x + 36 = 5$$

$$5x^2 - 30x + 45 - 5 = 0$$

$$5x^2 - 30x + 40 = 0$$

$$5(x^2 - 6x + 8) = 0$$

$$x^2 - 6x + 8 = 0$$

$$x^2 - 2x - 4x + 8 = 0$$

$$x(x-2) - 4(x-2) = 0$$

$$(x-2)(x-4) = 0$$

$$x - 2 = 0 \;\; or \;\; x - 4 = 0$$

$$x = 2 \;\; or \;\; x = 4$$

Putting $x = 2$ in (3)

$$y = 2(2) - 6$$

$$y = -2$$

Putting $x = 4$ in (3)

$$y = 2(4) - 6$$

$$y = 2$$

Hence solution set is $\{(2,-2),(4,2)\}$

Q#8: $\boldsymbol{(x+3)^2 + (y-1)^2 = 5;\; x^2 + y^2 + 2x = 9}$

Solution:

$(x+3)^2 + (y-1)^2 = 5 \qquad \longrightarrow (1)$

$x^2 + y^2 + 2x = 9 \qquad \longrightarrow (2)$

From equation (1)

$x^2 + 6x + 9 + y^2 - 2y + 1 = 5$

$x^2 + y^2 + 6x - 2y = -5 \quad \longrightarrow (3)$

Subtracting equations (2) and (3)

$$\begin{array}{rl} x^2 + y^2 + 6x - 2y &= -5 \\ x^2 + y^2 + 2x \qquad\;\; &= 9 \\ - \quad - \quad - \qquad\quad & - \\ \hline 4x - 2y &= -14 \end{array}$$

$2x - y = -7$

$$y = 2x + 7 \qquad \longrightarrow (4)$$

Putting value from equation (4) in (2)

$x^2 + (2x+7)^2 + 2x = 9$

$x^2 + 4x^2 + 28x + 49 + 2x = 9$

$5x^2 + 30x + 40 = 0$

$x^2 + 6x + 8 = 0$

$x^2 + 4x + 2x + 8 = 0$

$x(x+4) + 2(x+4) = 0$

$(x+4)(x+2) = 0$

$x + 4 = 0 \;\; or \;\; x + 2 = 0$

$x = -4 \;\; or \;\; x = -2$

Putting $x = -4$ in equation (4)

$$y = 2(-4) + 7$$

$$y = -8 + 7$$

$$y = -1$$

Putting $x = -2$ in equation (4)

$$y = 2(-2) + 7$$

$$y = -4 + 7$$

$$y = 3$$

Hence solution set is $\{(-4,-1),(-2,3)\}$

Q#9: $\boldsymbol{x^2 + (y+1)^2 = 18;\; (x+2)^2 + y^2 = 21}$

Solution:

$x^2 + (y+1)^2 = 18 \qquad \longrightarrow (1)$

$(x+2)^2 + y^2 = 21 \quad \longrightarrow (2)$

From equation (1)

$x^2 + y^2 + 2y + 1 = 18$

$x^2 + y^2 + 2y = 17 \quad \longrightarrow (3)$

From equation (2)

$x^2 + 4x + 4 + y^2 = 21$

$x^2 + y^2 + 4x = 17 \quad \longrightarrow (4)$

Subtracting equations (3) and (4)

$$\begin{array}{l} x^2 + y^2 + 2y = 17 \\ x^2 + y^2 + 4x = 17 \\ - \quad - \quad - \quad\quad - \\ \hline 2y - 4x = 0 \end{array}$$

$y - 2x = 0$

$$y = 2x \longrightarrow (5)$$

Putting value from equation (5) in (3)

$x^2 + (2x)^2 + 2(2x) = 17$

$x^2 + 4x^2 + 4x = 17$

$5x^2 + 4x - 17 = 0$

Using quadratic formula

$$x = \frac{-4 \pm \sqrt{(4)^2 - 4(5)(-17)}}{10}$$

$$x = \frac{-4 \pm \sqrt{16 + 340}}{10}$$

$$x = \frac{-4 \pm \sqrt{356}}{10}$$

$$x = \frac{-4 \pm 2\sqrt{89}}{10}$$

$$x = \frac{-2 \pm \sqrt{89}}{5}$$

Putting $x = \frac{-2+\sqrt{89}}{5}$ in equation (5)

$$y = 2\left(\frac{-2 + 2\sqrt{89}}{5}\right)$$

$$y = \frac{-4 + 2\sqrt{89}}{5}$$

Putting $x = \frac{-2-\sqrt{89}}{5}$ in equation (5)

$$y = 2\left(\frac{-2 - 2\sqrt{89}}{5}\right)$$

$$y = \frac{-4 - 2\sqrt{89}}{5}$$

Hence solution set is $\left\{\left(\frac{-2+\sqrt{89}}{5}, \frac{-4+2\sqrt{89}}{5}\right), \left(\frac{-2-\sqrt{89}}{5}, \frac{-4-2\sqrt{89}}{5}\right)\right\}$

Q#10: $\boldsymbol{x^2 + y^2 + 6x = 1;\ x^2 + y^2 + 2(x + y) = 3}$

Solution:

$x^2 + y^2 + 6x = 1 \quad \longrightarrow (1)$

$x^2 + y^2 + 2(x + y) = 3 \quad \longrightarrow (2)$

From equation (2)

$x^2 + y^2 + 2x + 2y = 3 \quad \longrightarrow (3)$

Subtracting equations (1) and (3)

$$\begin{array}{l} x^2 + y^2 + 2x + 2y = 3 \\ x^2 + y^2 + 6x \quad\quad = 1 \\ - \quad - \quad - \quad\quad\quad - \\ \hline -4x + 2y = 2 \end{array}$$

$-2x + y = 1$

$$y = 2x + 1 \quad \longrightarrow (4)$$

Putting value from equation (4) in (1)

$x^2 + (2x + 1)^2 + 6x = 1$

$x^2 + 4x^2 + 4x + 1 + 6x = 1$

$5x^2 + 10x = 0$

$5x(x + 2) = 0$

$5x = 0 \ \ or \ \ x + 2 = 0$

$x = 0 \ \ or \ \ x = -2$

Putting $x = 0$ in equation (4)

$$y = 2(0) + 1$$

$$y = 1$$

Putting $x = -2$ in equation (5)

$$y = 2(-2) + 1$$

$$y = -4 + 1$$

$$y = -3$$

Hence solution set is $\{(0, 1), (-2, -3)\}$