Exercise 4.7

Nature of roots of Quadratic Equation

We know that the roots of the quadratic equation $ax^2 + bx + c = 0$ are given by the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We see that there are two possible values for x, as discriminated by the part of the formula $\pm\sqrt{b^2 - 4ac}$. The nature of roots of the quadratic equation depends on the value of the expression $b^2 - 4ac$, which is called its discriminant.

Case I: If $b^2 - 4ac = 0$ then roots will be $\frac{-b}{2a}$ and $\frac{-b}{2a}$, so the roots are real and equal.

Case II: If $b^2 - 4ac < 0$ then roots will be complex and distinct.

Case III: If $b^2 - 4ac > 0$ then roots will be real and distinct. Also if If $b^2 - 4ac$ is a perfect square then roots will be rational otherwise irrational.

Q#1: Discuss the nature of roots of the following equations.

(i) $\boldsymbol{4x^2 + 6x + 1 = 0}$

Solution: $4x^2 + 6x + 1 = 0 \longrightarrow (1)$

$a = 4, b = 6, c = 1$

Now $b^2 - 4ac = 6^2 - 4(4)(1)$

$= 36 - 16 = 20 > 0$

$\Longrightarrow$ Roots of equation (1) will be real and distinct, also $b^2 - 4ac = 20$ is not a perfect square so roots will be irrational.

(ii) $\boldsymbol{x^2 - 5x + 6 = 0}$

Solution: $x^2 - 5x + 6 = 0 \longrightarrow (1)$

$a = 1, b = -5, c = 6$

Now $b^2 - 4ac = (-5)^2 - 4(1)(6)$

$= 25 - 24 = 1 > 0$

$\Longrightarrow$ Roots of equation (1) will be real and distinct, also $b^2 - 4ac = 1$ is a perfect square so roots will be rational.

(iii) $\boldsymbol{2x^2 - 5x + 1 = 0}$

Solution: $2x^2 - 5x + 1 = 0 \longrightarrow (1)$

$a = 2, b = -5, c = 1$

Now $b^2 - 4ac = (-5)^2 - 4(2)(1)$

$= 25 - 8 = 17 > 0$

$\Longrightarrow$ Roots of equation (1) will be real and distinct, also $b^2 - 4ac = 17$ is not a perfect square so roots will be irrational.

(iv) $\boldsymbol{25x^2 - 30x + 9 = 0}$

Solution: $25x^2 - 30x + 9 = 0 \longrightarrow (1)$

$a = 25, b = -30, c = 9$

Now $b^2 - 4ac = (-30)^2 - 4(25)(9)$

$= 900 - 900 = 0$

$\Longrightarrow$ Roots of equation (1) will be real and equal.

Q#2: Show that the roots of the following equations will be real.

(i) $\boldsymbol{x^2 - 2\left(m + \frac{1}{m}\right)x + 3 = 0}$

Solution: $x^2 - 2\left(m + \frac{1}{m}\right)x + 3 = 0 \longrightarrow (1)$

Equation (1) can be written as

$x^2 - 2\left(\frac{m^2 + 1}{m}\right)x + 3 = 0$

$mx^2 - 2(m^2 + 1)x + 3m = 0$

$a = m, b = -2(m^2 + 1), c = 3m$

Now $b^2 - 4ac = 4(m^2 + 1)^2 - 4(m)(3m)$

$= 4(m^4 + 2m^2 + 1) - 12m^2$

$= 4m^4 + 8m^2 + 4 - 12m^2$

$= 4m^4 - 4m^2 + 4$

$= 4(m^4 - m^2 + 1)$

Now as we know that $m^4 > m^2$

$\Longrightarrow m^4 + 1 > m^2$

$\Longrightarrow m^4 - m^2 + 1 > 0$

$\Longrightarrow 4(m^4 - m^2 + 1) > 0$

$\Longrightarrow b^2 - 4ac > 0$

$\Longrightarrow$ Roots of (1) will be real.

(ii) $\boldsymbol{(b - c)x^2 + (c - a)x + (a - b) = 0}$, $\boldsymbol{a, b, c \in Q}$

Solution:

$(b - c)x^2 + (c - a)x + (a - b) = 0 \longrightarrow (1)$

$A = (b - c), B = (c - a), C = (a - b)$

Now $B^2 - 4AC = (c - a)^2 - 4(b - c)(a - b)$

$= c^2 + a^2 - 2ac - 4(ab - b^2 - ac + bc)$

$= c^2 + a^2 - 2ac - 4ab + 4b^2 + 4ac - 4bc$
$= a^2 + 4b^2 + c^2 + 2ac - 4ab - 4bc$
$= (a - 2b + c)^2 > 0$
$\Longrightarrow$ Roots of (1) will be real.

Q#3: Show that the roots of the following equations will be rational.

(i) $(p+q)x^2 - px - q = 0 \dashrightarrow (1)$

Solution:

$$a = p+q, b = -p, c = -q$$

Now $b^2 - 4ac = (-p)^2 - 4(p+q)(-q)$
$= p^2 + 4q(p+q)$
$= p^2 + 4pq + q^2$
$= (p+2q)^2$

As $b^2 - 4ac > 0$ and also a perfect square, so roots of (1) are rational.

(ii) $px^2 - (p-q)x - q = 0 \dashrightarrow (1)$

Solution:

$$a = p, b = p - q, c = -q$$

Now $b^2 - 4ac = (p-q)^2 - 4(p)(-q)$
$= p^2 + q^2 - 2pq + 4pq$
$= p^2 + q^2 + 2pq$
$= (p+q)^2$

As $b^2 - 4ac > 0$ and also a perfect square, so roots of (1) are rational.

Q#4: For what value of m will the roots of the following equations be equal?

(i)
$(m+1)x^2 + 2(m+3)x + m + 8 = 0 \ \ \to (1)$

Solution:

$$a = m+1, b = 2(m+3), c = m+8$$

Since roots of equation (1) are equal
$\Longrightarrow\ b^2 - 4ac = 0$
$\Longrightarrow\ 4(m+3)^2 - 4(m+1)(m+8) = 0$
$\Longrightarrow\ (m+3)^2 - (m+1)(m+8) = 0$
$\Longrightarrow\ m^2 + 6m + 9 - (m^2 + 8m + m + 8) = 0$
$\Longrightarrow\ m^2 + 6m + 9 - (m^2 + 9m + 8) = 0$
$\Longrightarrow\ m^2 + 6m + 9 - m^2 - 9m - 8 = 0$
$\Longrightarrow\ -3m + 1 = 0$
$\Longrightarrow\ 3m = 1$
$\Longrightarrow\ m = \dfrac{1}{3}$

(ii) $x^2 - 2(1+3m)x + 7(3+2m) = 0 \dashrightarrow (1)$

Solution:

$$a = 1, b = -2(1+3m), c = 7(3+2m)$$

Since roots of equation (1) are equal
$\Longrightarrow\ b^2 - 4ac = 0$
$\Longrightarrow\ 4(1+3m)^2 - 4(1)(7(3+2m)) = 0$
$\Longrightarrow\ (1+3m)^2 - 7(3+2m) = 0$
$\Longrightarrow\ 1 + 9m^2 + 6m - 21 - 14m = 0$
$\Longrightarrow\ 9m^2 - 8m - 20 = 0$
$\Longrightarrow\ 9m^2 + 10m - 18m - 20 = 0$
$\Longrightarrow\ 9m(m+10) - 2(m+10) = 0$
$\Longrightarrow\ (m+10)(9m-2) = 0$
$\Longrightarrow\ m + 10 = 0\ \ or\ \ 9m - 2 = 0$
$\Longrightarrow\ m = -10\ \ or\ \ m = \dfrac{2}{9}$

(iii)
$(1+m)x^2 - 2(1+3m)x + (1+8m) = 0$

Solution:

$(1+m)x^2 - 2(1+3m)x + (1+8m) = 0 \dashrightarrow (1)$

$$a = 1+m, b = -2(1+3m), c = 1+8m$$

Since roots of (1) are equal
$\Longrightarrow\ b^2 - 4ac = 0$
$\Longrightarrow\ 4(1+3m)^2 - 4(1+m)(1+8m) = 0$
$\Longrightarrow\ (1+3m)^2 - (1+m)(1+8m) = 0$
$\Longrightarrow\ 1 + 9m^2 + 6m - (1 + 8m + m + 8m^2) = 0$
$\Longrightarrow\ 1 + 9m^2 + 6m - (1 + 9m + 8m^2) = 0$
$\Longrightarrow\ 1 + 9m^2 + 6m - 1 - 9m - 8m^2 = 0$
$\Longrightarrow\ m^2 - 3m = 0$
$\Longrightarrow\ m(m-3) = 0$
$\Longrightarrow\ m = 0, m - 3 = 0$
$\Longrightarrow\ m = 0, m = 3$

Q#5:
Show that the roots of $x^2 + (mx+c)^2 = a^2$ will be equal, if $c^2 = a^2(1+m^2)$.

Solution:

$$x^2 + (mx+c)^2 = a^2 \dashrightarrow (1)$$
$$x^2 + m^2x^2 + 2mcx + c^2 - a^2 = 0$$
$$x^2(1+m^2) + 2mcx + c^2 - a^2 = 0$$
$$A = 1+m^2, B = 2mc, C = c^2 - a^2$$

Since roots of (1) are equal
$\Longrightarrow\ B^2 - 4AC = 0$
$\Longrightarrow\ 4m^2c^2 - 4(1+m^2)(c^2 - a^2) = 0$
$\Longrightarrow\ m^2c^2 - (1+m^2)(c^2 - a^2) = 0$
$\Longrightarrow\ m^2c^2 - (c^2 - a^2 + m^2c^2 - m^2a^2) = 0$
$\Longrightarrow\ m^2c^2 - c^2 + a^2 - m^2c^2 + m^2a^2 = 0$
$\Longrightarrow\ -c^2 + a^2 + m^2a^2 = 0$
$\Longrightarrow\ c^2 = a^2(1+m^2)$ as required.

Q#6: Show that roots of $(mx+c)^2 = 4ax$ will be equal, if $c = \frac{a}{m}, m \neq 0$.

Solution:

$$(mx+c)^2 = 4ax \dashrightarrow (1)$$
$$m^2x^2 + 2mcx + c^2 - 4ax = 0$$
$$m^2x^2 + 2(mc - 2a)x + c^2 = 0$$

Since roots of the equation (1) are equal
$\Longrightarrow\ b^2 - 4ac = 0$
$\Longrightarrow\ 4(mc - 2a)^2 - 4m^2c^2 = 0$
$\Longrightarrow\ (mc - 2a)^2 - m^2c^2 = 0$
$\Longrightarrow\ (m^2c^2 - 4mca + 4a^2) - m^2c^2 = 0$
$\Longrightarrow\ m^2c^2 - 4mca + 4a^2 - m^2c^2 = 0$

$\Rightarrow\ -4mca + 4a^2 = 0$
$\Rightarrow\ -mc + a = 0$
$\Rightarrow\ c = \frac{a}{m}$ as required.

Q#7: Prove that $\boldsymbol{\frac{x^2}{a^2} + \frac{(mx+b)^2}{b^2} = 1}$ will have equal roots if $\boldsymbol{c^2 = a^2m^2 + b^2,\ a \neq 0, b \neq 0.}$

Solution: Do yourself

Q#8: Show that the roots of the equation $\boldsymbol{(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0}$ will be equal, if either $\boldsymbol{a^2 + b^2 + c^2 = 3abc\ or\ b = 0.}$

Solution: $(a^2 - bc)x^2 + 2(b^2 - ca)x + c^2 - ab = 0 \longrightarrow (1)$

Since roots of (1) are equal

$\Rightarrow\ b^2 - 4ac = 0$
$\Rightarrow\ 4(b^2 - ca)^2 - 4(a^2 - bc)(c^2 - ab) = 0$
$\Rightarrow\ (b^2 - ca)^2 - (a^2 - bc)(c^2 - ab) = 0$
$\Rightarrow\ b^4 + c^2a^2 - 2b^2ca - (a^2c^2 - a^3b - bc^3 + ab^2c) = 0$
$\Rightarrow\ b^4 + c^2a^2 - 2b^2ca - a^2c^2 + a^3b + bc^3 - ab^2c = 0$
$\Rightarrow\ b^4 - 2b^2ca + a^3b + bc^3 - ab^2c = 0$
$\Rightarrow (b^3 - 2bca + a^3 + c^3 - abc)b = 0$
$\Rightarrow (a^3 + b^3 + c^3 - 3abc)b = 0$
$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0\ or\ b = 0$
$\Rightarrow a^3 + b^3 + c^3 = 3abc\ or\ b = 0$ as required.