Exercise 3.4 (Solution)

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Q1.If $A=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}$ & $B=\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$ **then show that $A+B$ is symmetric.**

$$A=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}\quad B=\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$$

Let,

$$Z=A+B$$

$$Z=\begin{bmatrix}1&-2&5\\-2&3&-1\\5&-1&0\end{bmatrix}+\begin{bmatrix}-3&1&-2\\1&0&-1\\-2&-1&2\end{bmatrix}$$

$$Z=\begin{bmatrix}-2&-1&5\\-1&3&-1\\5&-1&0\end{bmatrix}\Rightarrow Z^t=\begin{bmatrix}-2&-1&5\\-1&3&-1\\5&-1&0\end{bmatrix}$$

$$Z^t=Z$$

So, Z is symmetric matrix.

Q2: If $A=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}$ **then show that,**

(i). $A+A^t$ is symmetric matrix.
(ii). $A-A^t$ is skew symmetric matrix.

$$A=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}$$

(i) Let,

$$Z=A+A^t$$

$$Z=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}+\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$$

$$Z=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}\Rightarrow Z^t=\begin{bmatrix}2&5&-1\\5&4&2\\-1&2&4\end{bmatrix}$$

$$Z^t=Z$$

So, Z is symmetric matrix.

(ii) Let,

$$Z=A-A^t$$

$$Z=\begin{bmatrix}1&2&0\\3&2&-1\\-1&3&2\end{bmatrix}-\begin{bmatrix}1&3&-1\\2&2&3\\0&-1&2\end{bmatrix}$$

$$Z=\begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}\Rightarrow Z^t=\begin{bmatrix}0&1&-1\\-1&0&4\\1&-4&0\end{bmatrix}$$

$$Z^t=-\begin{bmatrix}0&-1&1\\1&0&-4\\-1&4&0\end{bmatrix}\Rightarrow Z^t=-Z$$

So, Z is skew symmetric matrix.

Q3: If A is a square matrix of order 3 then show that,

(i). $A+A^t$ is symmetric matrix.
(ii). $A-A^t$ is skew symmetric matrix.

(i). Since A is square matrix of order 3, then we have to show that $A+A^t$ is symmetric matirix. Let,

$$Z=A+A^t$$
$$Z^t=\left(A+A^t\right)^t$$
$$Z^t=(A^t)^t+A$$

as $(A^t)^t=A$

$$Z^t=A+A^t$$
$$Z^t=Z$$

So, Z is symmetric matrix.

(ii). Since A is square matrix of order 3, then we have to show that $A-A^t$ is symmetric matirix. Let,

$$Z=A-A^t$$
$$Z^t=\left(A-A^t\right)^t$$
$$Z^t=-(A^t)^t+A^t$$

as $(A^t)^t=A$

$$Z^t=-A+A^t=-(A-A^t)$$
$$Z^t=-Z$$

So, Z is skew symmetric matrix.

Q4: If A and B are symmetric matrix and $AB=BA$. Show that AB is symmetric matrix.

Given that A and B are symmetric matrix then by the definition,

$$A^t=A;\ B^t=B$$

To show that AB is symmetric matrix we have,

$$(AB)^t=B^tA^t$$

Since $AB = BA$,

$$(AB)^t = BA$$

as given that, $AB = BA$,

$$(AB)^t = AB$$

hence AB is symmetric matrix.

Q5: Show that AA^t and A^tA are symmetric for any matrix of 2×3.

Let,

$$Z = AA^t$$

to show that Z is symmetric matrix we have,

$$Z^t = (AA^t)^t \Rightarrow Z^t = (A^t)^t A^t = AA^t$$

so, Z is symmetric matrix.

Now, let,

$$F = A^tA$$

to show that F is symmetric matrix we have,

$$F^t = (A^tA)^t \Rightarrow F^t = A^t(A^t)^t = A^tA$$

So, F is symmetric matrix.

Q6 If $A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$, show that,

(i). $A + (\bar{A})^t$ is Hermitian.

(i). $A - (\bar{A})^t$ is skew Hermitian.

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$$

(i). let,

$$Z = A + (\bar{A})^t$$

first,

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$Z = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} + \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix} = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$\bar{Z} = \begin{bmatrix} 0 & 2-i \\ 2+i & 0 \end{bmatrix} \Rightarrow (\bar{Z})^t = \begin{bmatrix} 0 & 2+i \\ 2-i & 0 \end{bmatrix}$$

$$(\bar{Z})^t = Z$$

So, Z is Hermitian matrix.

$$A = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix}$$

(ii). let,

$$Z = A - (\bar{A})^t$$

first,

$$\bar{A} = \begin{bmatrix} -i & 1-i \\ 1 & i \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix}$$

$$Z = \begin{bmatrix} i & 1+i \\ 1 & -i \end{bmatrix} - \begin{bmatrix} -i & 1 \\ 1-i & i \end{bmatrix} = \begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$\bar{Z} = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} \Rightarrow (\bar{Z})^t = \begin{bmatrix} -2i & -i \\ -i & 2i \end{bmatrix} = -\begin{bmatrix} 2i & i \\ i & -2i \end{bmatrix}$$

$$(\bar{Z})^t = -Z$$

So, Z is skew Hermitian matrix.

Q7: If A is symmetric or skew symmetric, show that A^2 is symmetric.

Given that A is symmetric matrix, then by the definition,

$$A^t = A$$

we have to show that A^2 is symmetric,

$$\left(A^2\right)^t = \left(A.A\right)^t = A^tA^t$$

$$\left(A^2\right)^t = A.A = A^2$$

So, A is symmetric matrix.

Now, consider A is skew matrix then by the definition,

$$A^t = -A$$

we have to show that A^2 is symmetric,

$$\left(A^2\right)^t = \left((-A).(-A)\right)^t = A^tA^t$$

$$\left(A^2\right)^t = A.A = A^2$$

So, A is symmetric matrix.

Q8: If $A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$ find $A(\bar{A})^t$.

$$A = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} 1 \\ 1-i \\ -i \end{bmatrix} \Rightarrow (\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \end{bmatrix}$$

Now,

$$A(\bar{A})^t = \begin{bmatrix} 1 \\ 1+i \\ i \end{bmatrix}\begin{bmatrix} 1 & 1-i & -i \end{bmatrix} = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 1-i^2 & -i-i^2 \\ i & i-i^2 & -i^2 \end{bmatrix}$$

$$A(\bar{A})^t = \begin{bmatrix} 1 & 1-i & -i \\ 1+i & 2 & 1-i \\ i & 1+i & 1 \end{bmatrix}$$

Q9: Find the inverse of the following matrices by the row operations.

(i).

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ -2 & -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_3 + 2R_1$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -2 & 0 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} R_3 + 2R_1$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 1 & 0 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 0 & 1 \end{bmatrix} -\frac{1}{2}R_2$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ R_3 - 2R_2 \end{matrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} -\frac{1}{4}R_3$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} R_1 + 3R_3$$

Hence, the inverse of the matrix is,

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

(ii).

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} R_3 - R_1$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & 5 \\ 0 & -1 & 3 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \begin{matrix} R_1 + 2R_2 \\ R_3 - 2R_2 \end{matrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ -1 & -2 & 1 \end{bmatrix} -R_2$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 0 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} -\frac{1}{3}R_2$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{matrix} R_1 - 5R_3 \\ R_2 + 3R_3 \end{matrix}$$

So, the inverse of the matrix is,

$$\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & \frac{5}{3} \\ 1 & 1 & -1 \\ \frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

(iii).

$$\begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & -3 & 2 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 7 & -4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} R_2 - 2R_1$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & -3 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} R_2 - 6R_3$$

$$\underset{\sim}{=R} \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 & 3 & 18 \\ -2 & 1 & 6 \\ -2 & 1 & 7 \end{bmatrix} \begin{matrix} R_1 + 3R_2 \\ R_3 + R_2 \end{matrix}$$

$$= \underset{\sim}{R}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}11 & -5 & -38\\ 2 & -1 & -8\\ -2 & 1 & 7\end{bmatrix}\begin{matrix}R_1-8R_3\\ R_2-2R_2\end{matrix}$$

So, the inverse of the matrix is,

$$\begin{bmatrix}11 & -5 & -38\\ 2 & -1 & -8\\ -2 & 1 & 7\end{bmatrix}$$

Q10: Find the rank of the following matrices.

(i).

$$A=\begin{bmatrix}1 & -1 & 2 & 1\\ 2 & -6 & 5 & 1\\ 3 & 5 & 4 & -3\end{bmatrix}$$

$$Rank\ of\ A=1+\begin{bmatrix}\begin{vmatrix}1 & -1\\ 2 & -6\end{vmatrix} & \begin{vmatrix}1 & 2\\ 2 & 5\end{vmatrix} & \begin{vmatrix}1 & 1\\ 2 & 1\end{vmatrix}\\ \begin{vmatrix}1 & -1\\ 3 & 5\end{vmatrix} & \begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix} & \begin{vmatrix}1 & 1\\ 3 & -3\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=1+\begin{bmatrix}-4 & 1 & -1\\ 8 & -2 & -6\end{bmatrix}$$

$$Rank\ of\ A=1+1+\begin{bmatrix}\begin{vmatrix}-4 & 1\\ 8 & -2\end{vmatrix} & \begin{vmatrix}-4 & -1\\ 8 & -6\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=2+\begin{bmatrix}0 & 32\end{bmatrix}=2+1$$

Remaining non-zero number of row is 1 so we write it 1.

$$Rank\ of\ A=3$$

(ii).

$$A=\begin{bmatrix}1 & -4 & -7\\ 2 & -5 & 1\\ 1 & -2 & 3\\ 3 & -7 & 4\end{bmatrix}$$

$$Rank\ of\ A=1+\begin{bmatrix}\begin{vmatrix}1 & -4\\ 2 & 5\end{vmatrix} & \begin{vmatrix}1 & -7\\ 2 & 1\end{vmatrix}\\ \begin{vmatrix}1 & -4\\ 1 & -2\end{vmatrix} & \begin{vmatrix}1 & -7\\ 1 & 3\end{vmatrix}\\ \begin{vmatrix}1 & -4\\ 3 & -7\end{vmatrix} & \begin{vmatrix}1 & -7\\ 3 & 4\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=1+\begin{bmatrix}3 & 15\\ 2 & 10\\ 5 & 25\end{bmatrix}$$

$$Rank\ of\ A=1+1+\begin{bmatrix}\begin{vmatrix}3 & 15\\ 2 & 10\end{vmatrix}\\ \begin{vmatrix}3 & 15\\ 5 & 25\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=2+\begin{bmatrix}0\\ 0\end{bmatrix}=2+0$$

Because remaining matrix is zero so we write it zero.

$$Rank\ of\ A=2$$

(iii).

$$A=\begin{bmatrix}3 & -1 & 3 & 0 & -1\\ 1 & 2 & -1 & -3 & -2\\ 2 & 3 & 4 & 2 & 5\\ 2 & 5 & -2 & -3 & 3\end{bmatrix}$$

$$Rank\ of\ A=1+\begin{bmatrix}\begin{vmatrix}3 & -1\\ 1 & 2\end{vmatrix} & \begin{vmatrix}3 & 3\\ 1 & -1\end{vmatrix} & \begin{vmatrix}3 & 0\\ 1 & -3\end{vmatrix} & \begin{vmatrix}3 & -1\\ 1 & -2\end{vmatrix}\\ \begin{vmatrix}3 & -1\\ 2 & 3\end{vmatrix} & \begin{vmatrix}3 & 3\\ 2 & 4\end{vmatrix} & \begin{vmatrix}3 & 0\\ 2 & 2\end{vmatrix} & \begin{vmatrix}3 & -1\\ 2 & 5\end{vmatrix}\\ \begin{vmatrix}3 & -1\\ 2 & 5\end{vmatrix} & \begin{vmatrix}3 & 3\\ 2 & -2\end{vmatrix} & \begin{vmatrix}3 & 0\\ 2 & -3\end{vmatrix} & \begin{vmatrix}3 & -1\\ 2 & 3\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=1+\begin{bmatrix}7 & -6 & -9 & -5\\ 11 & 6 & 6 & 17\\ 17 & -12 & -9 & 11\end{bmatrix}$$

$$Rank\ of\ A=1+1+\begin{bmatrix}\begin{vmatrix}7 & -6\\ 11 & 6\end{vmatrix} & \begin{vmatrix}7 & -9\\ 11 & 6\end{vmatrix} & \begin{vmatrix}7 & -5\\ 11 & 17\end{vmatrix}\\ \begin{vmatrix}7 & -6\\ 17 & -12\end{vmatrix} & \begin{vmatrix}7 & -9\\ 17 & -9\end{vmatrix} & \begin{vmatrix}7 & -5\\ 17 & 11\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=2+\begin{bmatrix}108 & 141 & 174\\ 18 & 90 & 213\end{bmatrix}$$

$$Rank\ of\ A=2+1+\begin{bmatrix}\begin{vmatrix}108 & 141\\ 18 & 90\end{vmatrix} & \begin{vmatrix}108 & 174\\ 18 & 213\end{vmatrix}\end{bmatrix}$$

$$Rank\ of\ A=3+\begin{bmatrix}7182 & 19872\end{bmatrix}=3+1$$

Remaining non-zero number of row is 1 so we write it 1.

$$Rank\ of\ A=4$$