

Solution. \emptyset no-element subsets

Conclusion. Above four examples show that, "every set has two improper subsets and remaining are proper subsets".

Example . Improper subsets = $\emptyset, \{x, y, z\}$

Proper subsets = $\{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, z\}$

Note that to find the number of subsets of a set having n elements, the formula is 2^n . For example, in

Example - 1. Number of subsets = $2^3 = 8$

Example - 2. Number of subsets = $2^2 = 4$

Example - 3. Number of subsets = $2^1 = 2$

Example - 4. Number of subsets = $2^0 = 1$

POWER SET. The set of all the subsets of a set is called its power set. It is written as $P(S)$. e.g.

Power set of $A = \{x, y, z\}$ above is

$$P(A) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, z\}, \{x, y, z\}\}$$

EXERCISE 2.1

1. Write the following sets in set-builder notation :

Solution.

(i) $\{1, 2, 3, \dots, 1000\} = \{x \mid x \in \mathbb{N} \wedge x \leq 1000\}$

(ii) $\{0, 1, 2, \dots, 100\} = \{x \mid x \in \mathbb{W} \wedge x \leq 100\}$

(iii) $\{0, \pm 1, \pm 2, \dots, \pm 1000\} = \{x \mid x \in \mathbb{Z} \wedge -1000 \leq x \leq 1000\}$

(iv) $\{0, -1, -2, \dots, -500\} = \{x \mid x \in \mathbb{Z} \wedge -500 \leq x \leq 0\}$

(v) $\{100, 101, 102, \dots, 400\} = \{x \mid x \in \mathbb{Z} \wedge 100 \leq x \leq 400\}$
 $= \{x \mid x \in \mathbb{N} \wedge 100 \leq x \leq 400\}$

(vi) $\{-100, -101, -102, \dots, -500\} = \{x \mid x \in \mathbb{Z} \wedge -500 \leq x \leq -100\}$

(vii) $\{\text{Peshawar, Lahore, Karachi, Quetta}\}$

$$= \{x \mid x \text{ is a capital of a province of Pakistan}\}$$

- (viii) { January, June, July }
 $= \{ x \mid x \text{ is a month of the Calendar year beginning with letter J} \}$
- (ix) The set of all odd natural numbers
 $= \{ x \mid x \text{ is an odd natural number} \}$
- (x) The set of all rational numbers. $= \{ x \mid x \in \mathbb{Q} \}$
- (xi) The set of all real numbers between 1 and 2 $= \{ x \mid x \in \mathbb{R} \wedge 1 < x < 2 \}$
- (xii) The set of all integers between -100 and 1000
 $= \{ x \mid x \in \mathbb{Z} \wedge -100 < x < 1000 \}$

2. Write each of the following sets in descriptive and tabular form:

Solution.

	<i>Descriptive Form</i> ↓	<i>Tabular Form</i> ↓
(i)	$\{ x \mid x \in \mathbb{N} \wedge x \leq 10 \}$ The set of first ten natural numbers	$\{ 1, 2, 3, \dots, 10 \}$
(ii)	$\{ x \mid x \in \mathbb{N} \wedge 4 < x < 12 \}$ The set of natural numbers between 4 and 12	$\{ 5, 6, 7, \dots, 11 \}$
(iii)	$\{ x \mid x \in \mathbb{Z} \wedge -5 < x < 5 \}$ The set of integers between -5 and 5	$\{ -4, -3, -2, \dots, 4 \}$
(iv)	$\{ x \mid x \in \mathbb{E} \wedge 2 < x \leq 4 \}$ The set of even integers between 2 and 5	$\{ 4 \}$
(v)	$\{ x \mid x \in \mathbb{P} \wedge x < 12 \}$ The set of prime numbers less than 12	$\{ 2, 3, 5, 7, 11 \}$
(vi)	$\{ x \mid x \in \mathbb{O} \wedge 3 < x < 12 \}$ The set of odd integers between 3 and 12	$\{ 5, 7, 9, 11 \}$
(vii)	$\{ x \mid x \in \mathbb{E} \wedge 4 \leq x \leq 10 \}$ The set of even integers between 2 and 12	$\{ 4, 6, 8, 10 \}$
(viii)	$\{ x \mid x \in \mathbb{E} \wedge 4 < x < 6 \}$ The set of even integers between 4 and 6	$\{ \}$
(ix)	$\{ x \mid x \in \mathbb{O} \wedge 5 \leq x \leq 7 \}$ The set of odd integers from 5 upto 7	$\{ 5, 7 \}$
(x)	$\{ x \mid x \in \mathbb{O} \wedge 5 < x < 7 \}$ The set of odd integers greater or equal 5 and less than 7	$\{ \}$
(xi)	$\{ x \mid x \in \mathbb{N} \wedge x + 4 = 0 \}$	

The set of natural numbers x satisfying $x + 4 = 0$

Tabular form : as $x + 4 = 0 \Rightarrow x = -4$ which $\notin \mathbb{N} \Rightarrow \{ \}$

(xii) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 2\}$

The set of rational numbers x satisfying $x^2 = 2$

Tabular form : as $x^2 = 2 \Rightarrow x = \sqrt{2}$ which $\notin \mathbb{Q} \Rightarrow \{ \}$

(xiii) $\{x \mid x \in \mathbb{R} \wedge x = x\}$

The set of real numbers x satisfying $x = x$

Tabular form : $x = x$ is satisfied by all reals. \Rightarrow not possible

(xiv) $\{x \mid x \in \mathbb{Q} \wedge x = -x\}$

The set of rational numbers x satisfying $x = -x$

Tabular form : $x = -x \Rightarrow x + x = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \Rightarrow \{0\}$

(xv) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$

The set of real numbers x satisfying $x \neq x$

Tabular form : $x \neq x$ as there is no real number which is not equal to itself $\Rightarrow \{ \}$

(xvi) $\{x \mid x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$

The set of real numbers which are not rational. not possible

Tabular form : set of reals is the union of rational & irrational numbers, so irrational $\Rightarrow \mathbb{Q}$

3. Which of the following sets are finite and which of these are infinite?

Solution.

- | | | |
|--------|---|--------------|
| (i) | The set of students of your class. | [Finite] |
| (ii) | The set of all schools in Pakistan. | [Finite] |
| (iii) | The set of natural numbers between 3 and 10. | [Finite] |
| (iv) | The set of rational numbers between 3 and 10. | [Infinite] |
| (v) | The set of real numbers between 0 and 1. | [Infinite] |
| (vi) | The set of rationals between 0 and 1. | [Infinite] |
| (vii) | The set of whole between 0 and 1. | [Finite] |
| (viii) | The set of all leaves of trees in Pakistan. | [Finite] |
| (ix) | $P(\mathbb{N})$ | [Infinite] |
| (x) | $P\{a, b, c\}$ | [Finite] |

- (xi) $\{1, 2, 3, 4, \dots\}$ [Infinite]
 (xii) $\{1, 2, 3, \dots, 100000000\}$ [Finite]
 (xiii) $\{x \mid x \in \mathbb{R} \wedge x \neq x\}$ [Finite]
 (xiv) $\{x \mid x \in \mathbb{R} \wedge x^2 = -16\}$ [Finite]
 (xv) $\{x \mid x \in \mathbb{Q} \wedge x^2 = 5\}$ [Finite]
 (xvi) $\{x \mid x \in \mathbb{Q} \wedge 0 \leq x \leq 1\}$ [Infinite]

4. Write two proper subsets of each of the following sets:

- (i) $\{a, b, c\}$ (ii) $\{0, 1\}$ (iii) N (iv) Z (v) Q
 (vi) R (vii) W (viii) $\{x \mid x \in \mathbb{Q} \wedge 0 < x \leq 2\}$

Solution.

- (i) Two proper subsets of $\{a, b, c\}$ are: $\{a\}, \{a, b\}$
 (ii) Two proper subsets of $\{0, 1\}$ are: $\{0\}, \{1\}$
 (iii) Two proper subsets of N are: $\{1\}, \{1, 2\}$
 (iv) Two proper subsets of Z are: $\{1\}, \{1, 2\}$
 (v) Two proper subsets of Q are: $\{1\}, \{1, 2\}$
 (vi) Two proper subsets of R are: $\{1\}, \{1, 2\}$
 (vii) Two proper subsets of W are: $\{1\}, \{1, 2\}$
 (viii) Two proper subsets of $\{x \mid x \in \mathbb{Q} \wedge 0 < x \leq 2\}$ are: $\{1\}, \{1, 2\}$

5. Is there any set which has no proper subset? If so, name the set.

Solution. Yes, empty set or $\{\}$ or \emptyset is the set which has no proper subset.

6. What is the difference between $\{a, b\}$ and $\{\{a, b\}\}$?

Solution.

$\{a, b\}$ is a set which contains two elements a and b .

and $\{\{a, b\}\}$ is a set which contains only one element $\{a, b\}$.

7. Which of the following sentences are true & which of them are false?

- (i) $\{1, 2\} = \{2, 1\}$ (ii) $\emptyset \subseteq \{\{a\}\}$ (iii) $\{a\} \subseteq \{\{a\}\}$
 (iv) $\{a\} \in \{\{a\}\}$ (v) $a \in \{\{a\}\}$ (vi) $\emptyset \in \{\{a\}\}$

Solution.

- (i) $\{1, 2\} = \{2, 1\}$ [True]
 (ii) $\emptyset \subseteq \{\{a\}\}$ [True]
 (iii) $\{a\} \subseteq \{\{a\}\}$ [False]

- (iv) $\{a\} \in \{\{a\}\}$ [True]
 (v) $a \in \{\{a\}\}$ [False]
 (vi) $\emptyset \in \{\{a\}\}$ [False]

8. What is the number of elements of the power set of each of the following sets?

- (i) $\{ \}$ (ii) $\{0, 1\}$ (iii) $\{1, 2, 3, 4, 5, 6, 7\}$
 (iv) $\{0, 1, 2, 3, 4, 5, 6, 7\}$ (v) $\{a, \{b, c\}\}$ (vi) $\{\{a, b\}, \{b, c\}, \{d, e\}\}$

Solution.

Note that formula to find the number of elements in power set is 2^n .

Number of elements in the

- (i) Power set of $\{ \}$ is $2^0 = 1$
 (ii) Power set of $\{0, 1\}$ is $2^2 = 4$
 (iii) Power set of $\{1, 2, 3, 4, 5, 6, 7\}$ is $2^7 = 128$
 (iv) Power set of $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is $2^8 = 256$
 (v) Power set of $\{a, \{b, c\}\}$ is $2^2 = 4$
 (vi) Power set of $\{\{a, b\}, \{b, c\}, \{d, e\}\}$ is $2^3 = 8$

9. Write down the power set of each of the following sets:

- (i) $\{9, 11\}$ (ii) $\{+, -, \times, \div\}$ (iii) $\{\emptyset\}$ (iv) $\{a, \{b, c\}\}$

Solution.

- (i) Power set of $\{9, 11\}$ is: $\{\emptyset, \{9\}, \{11\}, \{9, 11\}\}$
 (ii) Power set of $\{+, -, \times, \div\}$ is $\{\emptyset, \{+\}, \{-\}, \{\times\}, \{+\}, \{+,-\}, \{+, \times\}, \{+, \div\}, \{-, \times\}, \{-, \div\}, \{\times, \div\}, \{+, -, \times\}, \{+, -, \div\}, \{+, \times, \div\}, \{-, \times, \div\}, \{+,-, \times, \div\}\}$
 (iii) Power set of $\{\emptyset\}$ is: $\{\emptyset, \{\emptyset\}\}$
 (iv) Power set of $\{a, \{b, c\}\}$ is: $\{\emptyset, \{a\}, \{\{b, c\}\}, \{a, \{b, c\}\}\}$

10. Which of the pairs of sets are equivalent? Which of them are also equal?

Solution.

- (i) $\{a, b, c\}$, $\{1, 2, 3\}$
 are equivalent sets (since, each has three elements)
 (ii) The set of first 10 whole numbers; $\{0, 1, 2, \dots, 9\}$
 are equal sets (since, each has same ten elements)

- (iii) The set of angles of a quadrilateral $ABCD$;
 set of the sides of the same quadrilateral.
 are equivalent sets (since, each has four elements)
- (iv) Set of the sides of a hexagon $ABCDEF$;
 set of the angles of the same hexagon.
 are equivalent sets (since, each has six elements)
- (v) $\{1, 2, 3, 4, \dots\}$; $\{2, 4, 6, 8, \dots\}$
 are equivalent sets (since, 1-1 correspondence can be established)
- (vi) $\{1, 2, 3, 4, \dots\}$; $\left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$
 are equivalent sets (since, 1-1 correspondence can be established)
- (vii) $\{5, 10, 15, 20, \dots, 55555\}$; $\{5, 10, 15, 20, \dots\}$
 are not equivalent sets
 (since, first set has finite and second infinite number of elements).

§ 2.2 OPERATIONS ON SETS

UNIVERSAL SET. The set of all objects under consideration is called the universal Set. It is usually denoted by U . Any universal set can be restricted to a lower set according to the situation. e.g.,

U = set of all natural numbers

$U = \{1, 2, 3, \dots, 100\}$

$U = \{a, b, c, \dots, x, y, z\}$, etc.

OPERATIONS ON SETS.

Union of Two Sets. The union of sets A and B , denoted by $A \cup B$, is a set whose elements are the elements of A or the elements of B . In set builder form,

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \quad \text{OR}$$

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

For example, if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, then

$$A \cup B = \{1, 2, 3, 4, 5, 6\}.$$