

IMPORTANT FORMULAS

The Law of Cosine

$$\begin{aligned} \bullet a^2 &= b^2 + c^2 - 2bc \cos \alpha & \bullet b^2 &= c^2 + a^2 - 2ca \cos \beta & \bullet c^2 &= a^2 + b^2 - 2ab \cos \gamma \\ \bullet \cos \alpha &= \frac{b^2 + c^2 - a^2}{2bc} & \bullet \cos \beta &= \frac{c^2 + a^2 - b^2}{2ca} & \bullet \cos \gamma &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

The Law of Sine

$$\bullet \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

The Law of Tangent

$$\bullet \frac{a-b}{a+b} = \frac{\tan \frac{\alpha-\beta}{2}}{\tan \frac{\alpha+\beta}{2}} \quad \bullet \frac{b-c}{b+c} = \frac{\tan \frac{\beta-\gamma}{2}}{\tan \frac{\beta+\gamma}{2}} \quad \bullet \frac{c-a}{c+a} = \frac{\tan \frac{\gamma-\alpha}{2}}{\tan \frac{\gamma+\alpha}{2}}$$

Half Angle Formulas

$$\begin{aligned} \bullet \sin \frac{\alpha}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} & \bullet \sin \frac{\beta}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}} & \bullet \sin \frac{\gamma}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}} \\ \bullet \cos \frac{\alpha}{2} &= \sqrt{\frac{s(s-a)}{bc}} & \bullet \cos \frac{\beta}{2} &= \sqrt{\frac{s(s-b)}{ac}} & \bullet \cos \frac{\gamma}{2} &= \sqrt{\frac{s(s-c)}{ab}} \\ \bullet \tan \frac{\alpha}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} & \bullet \tan \frac{\beta}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}} & \bullet \tan \frac{\gamma}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \end{aligned}$$

$$\text{where } s = \frac{a+b+c}{2} \Rightarrow 2s = a+b+c$$

Area of the Triangle (= Δ)

$$\begin{aligned} \bullet \Delta &= \frac{1}{2}bc \sin \alpha = \frac{1}{2}ca \sin \beta = \frac{1}{2}ab \sin \gamma \\ \bullet \Delta &= \frac{a^2 \sin \beta \sin \gamma}{2 \sin \alpha} = \frac{b^2 \sin \gamma \sin \alpha}{2 \sin \beta} = \frac{c^2 \sin \alpha \sin \beta}{2 \sin \gamma} \\ \bullet \Delta &= \sqrt{s(s-a)(s-b)(s-c)} \quad (\text{Hero's Formula}) \end{aligned}$$

Circum Radius (= R)

$$\bullet R = \frac{a}{2 \sin \alpha} = \frac{b}{2 \sin \beta} = \frac{c}{2 \sin \gamma} \quad \bullet R = \frac{abc}{4\Delta}$$

In-Radius (= r)

$$\bullet r = \frac{\Delta}{s}$$

Escribed Circle

$$\bullet r_1 = \frac{\Delta}{s-a} \quad \bullet r_2 = \frac{\Delta}{s-b} \quad \bullet r_3 = \frac{\Delta}{s-c}$$

Question # 1

Show that

$$(i) r = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \quad (ii) s = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution

$$\begin{aligned} (i) \quad \text{R.H.S} &= 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \\ &= 4R \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \\ &= 4R \sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{(bc)(ac)(ab)}} \\ &= 4R \sqrt{\frac{(s-a)^2 (s-b)^2 (s-c)^2}{a^2 b^2 c^2}} \\ &= 4R \frac{(s-a)(s-b)(s-c)}{abc} \\ &= 4 \left(\frac{abc}{4\Delta} \right) \frac{(s-a)(s-b)(s-c)}{abc} \quad \because R = \frac{abc}{4\Delta} \\ &= \frac{(s-a)(s-b)(s-c)}{\Delta} = \frac{s(s-a)(s-b)(s-c)}{s\Delta} \\ &= \frac{\Delta^2}{s\Delta} \quad \because \Delta = \sqrt{s(s-a)(s-b)(s-c)} \\ &= \frac{\Delta}{s} = r = \text{L.H.S} \end{aligned}$$

$$\begin{aligned} (ii) \quad \text{R.H.S} &= 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \\ &= 4R \sqrt{\frac{s(s-a)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}} \\ &= 4R \sqrt{\frac{s^2 \cdot s(s-a)(s-b)(s-c)}{(bc)(ac)(ab)}} = 4R \sqrt{\frac{s^2 \Delta^2}{a^2 b^2 c^2}} \\ &= 4R \frac{s\Delta}{abc} = 4 \left(\frac{abc}{4\Delta} \right) s \frac{\Delta}{abc} \quad \because R = \frac{abc}{4\Delta} \\ &= s = \text{L.H.S} \end{aligned}$$

Question # 2

Show that:

$$r = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} = c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$$

Solution

We take

$$\begin{aligned}
 a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} &= a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \frac{1}{\cos \frac{\alpha}{2}} \\
 &= a \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \frac{1}{\sqrt{\frac{s(s-a)}{bc}}} \\
 &= a \sqrt{\frac{(s-a)(s-c)}{ac}} \sqrt{\frac{(s-a)(s-b)}{ab}} \sqrt{\frac{bc}{s(s-a)}} \\
 &= a \sqrt{\frac{(s-a)(s-c)(s-a)(s-b)(bc)}{(ac)(ab)s(s-a)}} \\
 &= a \sqrt{\frac{(s-a)(s-b)(s-c)}{a^2 s}} = a \sqrt{\frac{s(s-a)(s-b)(s-c)}{a^2 s^2}} \\
 &= a \frac{\sqrt{s(s-a)(s-b)(s-c)}}{as} = \frac{\Delta}{s} = r
 \end{aligned}$$

$$\Rightarrow a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} = r \dots\dots\dots (i)$$

Similarly prove yourself

$$b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} = r \dots\dots\dots (ii)$$

$$c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2} = r \dots\dots\dots (iii)$$

From (i), (ii) and (iii)

$$r = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \sec \frac{\alpha}{2} = b \sin \frac{\gamma}{2} \sin \frac{\alpha}{2} \sec \frac{\beta}{2} = c \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sec \frac{\gamma}{2}$$

Question # 3

Show that:

$$(i) r_1 = 4R \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \quad (ii) r_2 = 4R \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\gamma}{2}$$

$$(iii) r_3 = 4R \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\gamma}{2}$$

Solution

$$\begin{aligned}
 \text{R.H.S} &= 4R \sin \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \\
 &= 4R \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-b)}{ac}} \sqrt{\frac{s(s-c)}{ab}} \\
 &= 4R \sqrt{\frac{(s-b)(s-c)s(s-b)s(s-c)}{(bc)(ac)(ab)}} = 4R \sqrt{\frac{s^2 (s-b)^2 (s-c)^2}{a^2 b^2 c^2}}
 \end{aligned}$$

$$\begin{aligned}
 &= 4R \frac{s(s-b)(s-c)}{abc} = 4 \frac{abc}{4\Delta} \frac{s(s-b)(s-c)}{abc} \cdot \frac{(s-a)}{(s-a)} \quad \because R = \frac{abc}{4\Delta} \\
 &= \frac{s(s-a)(s-b)(s-c)}{\Delta(s-a)} \\
 &= \frac{\Delta^2}{\Delta(s-a)} = \frac{\Delta}{(s-a)} = r_1 = \text{R.H.S}
 \end{aligned}$$

(ii) & (iii) Do yourself

Question # 4

Show that:

$$(i) r_1 = s \tan \frac{\alpha}{2} \quad (ii) r_2 = s \tan \frac{\beta}{2} \quad (iii) r_3 = s \tan \frac{\gamma}{2}$$

Solution

$$\begin{aligned}
 \text{R.H.S} &= s \tan \frac{\alpha}{2} \\
 &= s \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = s \sqrt{\frac{(s-b)(s-c)}{s(s-a)} \cdot \frac{s(s-a)}{s(s-a)}} \\
 &= s \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-a)^2}} \\
 &= s \sqrt{\frac{\Delta^2}{s^2(s-a)^2}} = s \frac{\Delta}{s(s-a)} = \frac{\Delta}{(s-a)} = r_1 = \text{L.H.S}
 \end{aligned}$$

(ii) & (iii) Do yourself

Question # 5

Prove that:

$$\begin{aligned}
 (i) \quad r_1 r_2 + r_2 r_3 + r_3 r_1 &= s^2 & (ii) \quad r r_1 r_2 r_3 &= \Delta^2 \\
 (iii) \quad r_1 + r_2 + r_3 - r &= 4R & (iv) \quad r_1 r_2 r_3 &= r s^2
 \end{aligned}$$

Solution

$$\begin{aligned}
 (i) \quad \text{L.H.S} &= r_1 r_2 + r_2 r_3 + r_3 r_1 \\
 &= \left(\frac{\Delta}{s-a}\right)\left(\frac{\Delta}{s-b}\right) + \left(\frac{\Delta}{s-b}\right)\left(\frac{\Delta}{s-c}\right) + \left(\frac{\Delta}{s-c}\right)\left(\frac{\Delta}{s-a}\right) \\
 &= \frac{\Delta^2}{(s-a)(s-b)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-c)(s-a)} \\
 &= \Delta^2 \left(\frac{1}{(s-a)(s-b)} + \frac{1}{(s-b)(s-c)} + \frac{1}{(s-c)(s-a)} \right) \\
 &= \Delta^2 \left(\frac{s-c+s-a+s-b}{(s-a)(s-b)(s-c)} \right) = \Delta^2 \left(\frac{3s-(a+b+c)}{(s-a)(s-b)(s-c)} \right) \\
 &= \Delta^2 \left(\frac{3s-2s}{(s-a)(s-b)(s-c)} \right) \quad \because s = \frac{a+b+c}{2}
 \end{aligned}$$

$$\begin{aligned}
&= \Delta^2 \left(\frac{s}{(s-a)(s-b)(s-c)} \cdot \frac{s}{s} \right) \\
&= \Delta^2 \left(\frac{s^2}{s(s-a)(s-b)(s-c)} \right) = \Delta^2 \left(\frac{s^2}{\Delta^2} \right) = s^2 = \text{R.H.S}
\end{aligned}$$

(ii) L.H.S = $r_1 r_2 r_3$

$$\begin{aligned}
&= \left(\frac{\Delta}{s} \right) \left(\frac{\Delta}{s-a} \right) \left(\frac{\Delta}{s-b} \right) \left(\frac{\Delta}{s-c} \right) \\
&= \frac{\Delta^4}{s(s-a)(s-b)(s-c)} = \frac{\Delta^4}{\Delta^2} = \Delta^2 = \text{R.H.S}
\end{aligned}$$

(iii) L.H.S = $r_1 + r_2 + r_3 - r$

$$\begin{aligned}
&= \frac{\Delta}{s-a} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} - \frac{\Delta}{s} = \Delta \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} \right) \\
&= \Delta \left(\frac{(s-b) + (s-a)}{(s-a)(s-b)} + \frac{s - (s-c)}{s(s-c)} \right) = \Delta \left(\frac{2s-b-a}{(s-a)(s-b)} + \frac{s-s+c}{s(s-c)} \right) \\
&= \Delta \left(\frac{a+b+c-b-a}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right) \quad \because 2s = a+b+c \\
&= \Delta \left(\frac{c}{(s-a)(s-b)} + \frac{c}{s(s-c)} \right) = c\Delta \left(\frac{1}{(s-a)(s-b)} + \frac{1}{s(s-c)} \right) \\
&= c\Delta \left(\frac{s(s-c) - (s-a)(s-b)}{s(s-a)(s-b)(s-c)} \right) = c\Delta \left(\frac{s^2 - sc + s^2 - as - bs + ab}{\Delta^2} \right) \\
&= c \left(\frac{2s^2 - s(a+b+c) + ab}{\Delta} \right) = c \left(\frac{2s^2 - s(2s) + ab}{\Delta} \right) \\
&= c \left(\frac{2s^2 - 2s^2 + ab}{\Delta} \right) = \frac{abc}{\Delta} = 4 \cdot \frac{abc}{4\Delta} = 4R = \text{R.H.S}
\end{aligned}$$

(iv) L.H.S = $\left(\frac{\Delta}{s-a} \right) \left(\frac{\Delta}{s-b} \right) \left(\frac{\Delta}{s-c} \right)$

$$\begin{aligned}
&= \frac{\Delta^3}{(s-a)(s-b)(s-c)} = \frac{s\Delta^3}{s(s-a)(s-b)(s-c)} \\
&= \frac{s\Delta^3}{\Delta^2} = s\Delta = s^2 \frac{\Delta}{s} = s^2 r = rs^2 = \text{R.H.S}
\end{aligned}$$

Question # 6

Find R, r, r_1, r_2 and r_3 , if measures of the sides of triangle ABC are

(i) $a=13$, $b=14$, $c=15$

(ii) $a=34$, $b=20$, $c=42$

Solution

(i) $a=13$, $b=14$, $c=15$

$$s = \frac{a+b+c}{2} = \frac{13+14+15}{2} = 21$$

$$s - a = 21 - 13 = 8$$

$$s - b = 21 - 14 = 7$$

$$s - c = 21 - 15 = 6$$

So $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$
 $= \sqrt{21(8)(7)(6)} = \sqrt{7056} = 84$

Now

$$R = \frac{abc}{4\Delta} = \frac{(13)(14)(15)}{4(84)} = 8.125$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

$$r_1 = \frac{\Delta}{s-a} = \frac{84}{8} = 10.5$$

$$r_2 = \frac{\Delta}{s-b} = \frac{84}{7} = 12$$

$$r_3 = \frac{\Delta}{s-c} = \frac{84}{6} = 14$$

(ii) *Do yourself*

Question # 7

Prove that in an equilateral triangle,

(i) $r : R : r_1 = 1 : 2 : 3$

(ii) $r : R : r_1 : r_2 : r_3 = 1 : 2 : 3 : 3 : 3$

Solution

(i) *Do yourself*

(ii) In equilateral triangle all the sides are equal so $a = b = c$

$$s = \frac{a+b+c}{2} = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$s - a = \frac{3a}{2} - a = \left(\frac{3}{2} - 1\right)a = \frac{1}{2}a$$

Now

$$\begin{aligned} \Delta &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{s(s-a)(s-a)(s-a)} = \sqrt{s(s-a)^3} = \sqrt{\frac{3a}{2} \left(\frac{1}{2}a\right)^3} \\ &= \sqrt{\frac{3a}{2} \left(\frac{a^3}{8}\right)} = \sqrt{\frac{3a^4}{16}} = \frac{\sqrt{3}a^2}{4} \end{aligned}$$

Now $r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}a^2}{4}}{\frac{3a}{2}} = \frac{\sqrt{3}a^2}{4} \cdot \frac{2}{3a} = \frac{\sqrt{3}a}{6}$

$$R = \frac{abc}{4\Delta} = \frac{a \cdot a \cdot a}{4\sqrt{3}a^2/4} = \frac{a}{\sqrt{3}} = \frac{a}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}a}{3}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{\sqrt{3}a^2/4}{\frac{1}{2}a} = \frac{\sqrt{3}a^2}{4} \cdot \frac{2}{a} = \frac{\sqrt{3}a}{2}$$

$$r_2 = \frac{\Delta}{s-b} = \frac{\Delta}{s-a} = \frac{\sqrt{3}a}{2}$$

$$r_3 = \frac{\Delta}{s-c} = \frac{\Delta}{s-a} = \frac{\sqrt{3}a}{2}$$

Now

$$\begin{aligned} r : R : r_1 : r_2 : r_3 &= \frac{\sqrt{3}a}{6} : \frac{\sqrt{3}a}{3} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2} : \frac{\sqrt{3}a}{2} \\ &= \frac{1}{6} : \frac{1}{3} : \frac{1}{2} : \frac{1}{2} : \frac{1}{2} && \div \text{ing by } \sqrt{3}a \\ &= 1 : 2 : 3 : 3 : 3 && \times \text{ing by } 6 \end{aligned}$$

Question # 8

Prove that:

$$(i) \Delta = r^2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \qquad (ii) r = s \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$$

$$(iii) \Delta = 4Rr \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$$

Solution

$$\begin{aligned} (i) \text{ R.H.S} &= r^2 \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2} \\ &= r^2 \frac{1}{\tan \frac{\alpha}{2}} \cdot \frac{1}{\tan \frac{\beta}{2}} \cdot \frac{1}{\tan \frac{\gamma}{2}} \\ &= r^2 \frac{1}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}} \cdot \frac{1}{\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}} \cdot \frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}} \\ &= r^2 \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} \cdot \sqrt{\frac{s(s-b)}{(s-a)(s-c)}} \cdot \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} \\ &= r^2 \sqrt{\frac{s^3(s-a)(s-b)(s-c)}{(s-a)^2(s-b)^2(s-c)^2}} = r^2 \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)}} \\ &= r^2 \sqrt{\frac{s^3}{(s-a)(s-b)(s-c)} \cdot \frac{s}{s}} = r^2 \sqrt{\frac{s^4}{s(s-a)(s-b)(s-c)}} \\ &= r^2 \sqrt{\frac{s^4}{\Delta^2}} = r^2 \frac{s^2}{\Delta} = \left(\frac{\Delta}{s}\right)^2 \frac{s^2}{\Delta} \qquad \because r = \frac{\Delta}{s} \end{aligned}$$

$$= \frac{\Delta^2}{s^2} \frac{s^2}{\Delta} = \Delta = \text{L.H.S}$$

(ii) *Do yourself*

(iii) R.H.S = $4Rr \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}$

$$= 4Rr \sqrt{\frac{s(s-a)}{bc}} \cdot \sqrt{\frac{s(s-b)}{ca}} \cdot \sqrt{\frac{s(s-c)}{ab}}$$

$$= 4Rr \sqrt{\frac{s(s-a) \cdot s(s-b) \cdot s(s-c)}{(bc)(ac)(ab)}}$$

$$= 4Rr \sqrt{\frac{s^2 \cdot s(s-a)(s-b)(s-c)}{a^2 b^2 c^2}} = 4Rr \sqrt{\frac{s^2 \cdot \Delta^2}{a^2 b^2 c^2}}$$

$$= 4Rr \frac{s\Delta}{abc} = 4 \left(\frac{abc}{4\Delta} \right) \left(\frac{\Delta}{s} \right) \frac{s\Delta}{abc} = \Delta = \text{L.H.S}$$

Question # 9

Show that

(i) $\frac{1}{2rR} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$

(ii) $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

Solution

(i) L.H.S = $\frac{1}{2rR}$

$$= \frac{1}{2 \left(\frac{\Delta}{s} \right) \left(\frac{abc}{4\Delta} \right)} = \frac{4s\Delta}{2\Delta abc} = \frac{2s}{abc} = \frac{a+b+c}{abc} \quad \because 2s = a+b+c$$

$$= \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc}$$

$$= \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \text{R.H.S}$$

(ii) R.H.S = $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$

$$= \frac{1}{\frac{\Delta}{s-a}} + \frac{1}{\frac{\Delta}{s-b}} + \frac{1}{\frac{\Delta}{s-c}} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{s-a+s-b+s-c}{\Delta} = \frac{3s-(a+b+c)}{\Delta}$$

$$= \frac{3s-2s}{\Delta} \quad \because 2s = a+b+c$$

$$= \frac{s}{\Delta} = \frac{1}{\frac{\Delta}{s}} = \frac{1}{r} = \text{L.H.S}$$

Question # 10

$$\text{We take } \frac{a \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos \frac{\alpha}{2}} = a \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \frac{1}{\cos \frac{\alpha}{2}}$$

Solution

Now see Question # 2

Question # 11Prove that: $abc(\sin \alpha + \sin \beta + \sin \gamma) = 4\Delta s$ **Solution**

$$\text{L.H.S} = abc(\sin \alpha + \sin \beta + \sin \gamma)$$

$$\text{Since } \Delta = \frac{1}{2}ab \sin \gamma = \frac{1}{2}bc \sin \alpha = \frac{1}{2}ca \sin \beta$$

$$\therefore \sin \gamma = \frac{2\Delta}{ab}, \quad \sin \alpha = \frac{2\Delta}{bc}, \quad \sin \beta = \frac{2\Delta}{ca}$$

$$\begin{aligned} \text{Thus L.H.S} &= abc \left(\frac{2\Delta}{bc} + \frac{2\Delta}{ac} + \frac{2\Delta}{ab} \right) \\ &= abc \left(\frac{2\Delta a + 2\Delta b + 2\Delta c}{abc} \right) = 2\Delta a + 2\Delta b + 2\Delta c \\ &= 2\Delta(a + b + c) = 2\Delta(2s) \quad \because 2s = a + b + c \\ &= 4\Delta s = \text{R.H.S} \end{aligned}$$

Question # 12

$$(i) (r_1 + r_2) \tan \frac{\gamma}{2} = c$$

$$(ii) (r_3 - r) \cot \frac{\gamma}{2} = c$$

Solution

$$\begin{aligned} (i) \quad \text{L.H.S} &= (r_1 + r_2) \tan \frac{\gamma}{2} \\ &= \left(\frac{\Delta}{s-a} + \frac{\Delta}{s-b} \right) \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \\ &= \left(\frac{\Delta(s-b) + \Delta(s-a)}{(s-a)(s-b)} \right) \sqrt{\frac{(s-a)(s-b)}{s(s-c)} \cdot \frac{s(s-c)}{s(s-c)}} \\ &= \Delta \left(\frac{s-b+s-a}{(s-a)(s-b)} \right) \sqrt{\frac{s(s-a)(s-b)(s-c)}{s^2(s-c)^2}} \\ &= \Delta \left(\frac{2s-a-b}{(s-a)(s-b)} \right) \sqrt{\frac{\Delta^2}{s^2(s-c)^2}} \\ &= \Delta \left(\frac{a+b+c-a-b}{(s-a)(s-b)} \right) \frac{\Delta}{s(s-c)} \quad \because 2s = a+b+c \\ &= \frac{\Delta^2 c}{s(s-a)(s-b)(s-c)} = \frac{\Delta^2 c}{\Delta^2} = c = \text{R.H.S} \end{aligned}$$

$$\begin{aligned}
\text{(ii) L.H.S} &= (r_3 - r) \cot \frac{\gamma}{2} \\
&= \left(\frac{\Delta}{s-c} - \frac{\Delta}{s} \right) \frac{1}{\tan \frac{\gamma}{2}} = \Delta \left(\frac{1}{s-c} - \frac{1}{s} \right) \frac{1}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}} \\
&= \Delta \left(\frac{s-(s-c)}{s(s-c)} \right) \sqrt{\frac{s(s-c)}{(s-a)(s-b)}} = \Delta \left(\frac{c}{s(s-c)} \right) \sqrt{\frac{s(s-c)}{(s-a)(s-b)} \cdot \frac{s(s-c)}{s(s-c)}} \\
&= \Delta \left(\frac{c}{s(s-c)} \right) \sqrt{\frac{s^2(s-c)^2}{s(s-a)(s-b)(s-c)}} = \Delta \left(\frac{c}{s(s-c)} \right) \sqrt{\frac{s^2(s-c)^2}{\Delta^2}} \\
&= \Delta \left(\frac{c}{s(s-c)} \right) \frac{s(s-c)}{\Delta} = c = \text{R.H.S}
\end{aligned}$$

If you found any error, submit at
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Error Analyst
Muhammad Tayyab (CTI): Govt Degree College Daska (2014-15) Saad - Punjab College (2016)

Book: **Exercise 12.8 (Page 385)**
Text Book of Algebra and Trigonometry Class XI
Punjab Textbook Board, Lahore.

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