

Question # 1

Prove that

(i) $\sin(180^\circ + \theta) = -\sin \theta$

(ii) $\cos(180^\circ + \theta) = -\cos \theta$

(iii) $\tan(270^\circ - \theta) = \cot \theta$

(iv) $\cos(\theta - 180^\circ) = -\cos \theta$

(v) $\cos(270^\circ + \theta) = \sin \theta$

(vi) $\sin(\theta + 270^\circ) = -\cos \theta$

(vii) $\tan(180^\circ + \theta) = \tan \theta$

(viii) $\cos(360^\circ - \theta) = \cos \theta$

Solution

$$(i) \text{ L.H.S} = \sin(180 + \theta) = \sin 180 \cos \theta + \cos 180 \sin \theta \\ = \sin(0)\cos \theta + (-1)\sin \theta = 0 - \sin \theta = -\sin \theta = \text{R.H.S}$$

(ii) *Do yourself*

$$(iii) \text{ L.H.S} = \tan(270^\circ - \theta) = \frac{\tan 270^\circ - \tan \theta}{1 + \tan 270^\circ \tan \theta} \\ = \frac{\tan 270^\circ \left(1 - \frac{\tan \theta}{\tan 270^\circ}\right)}{\tan 270^\circ \left(\frac{1}{\tan 270^\circ} + \tan \theta\right)} = \frac{\left(1 - \frac{\tan \theta}{\infty}\right)}{\left(\frac{1}{\infty} + \tan \theta\right)} \\ = \frac{(1-0)}{(0 + \tan \theta)} = \frac{1}{\tan \theta} = \cot \theta = \text{R.H.S}$$

*Remaining do yourself.***Question # 2**

Find the values of the following:

(i) $\sin 15^\circ$

(ii) $\cos 15^\circ$

(iii) $\tan 15^\circ$

Solution

(i) Since $15 = 45 - 30$

So $\sin 15^\circ = \sin(45 - 30) = \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

(ii) Since $15 = 45 - 30$

So $\cos 15^\circ = \cos(45 - 30) = \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$

$$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

(iii) Since $15 = 45 - 30$

$$\text{So } \tan 15^\circ = \tan(45 - 30) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + (1)\left(\frac{1}{\sqrt{3}}\right)} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}.$$

For (iv), (v) and (vi), we have hint:

Hint: Use $105 = 60 + 45$ in these questions

Question # 3

Prove that:

(i) $\sin(45^\circ + \alpha) = \frac{1}{\sqrt{2}}(\sin \alpha + \cos \alpha)$ (ii) $\cos(45^\circ + \alpha) = \frac{1}{\sqrt{2}}(\cos \alpha - \sin \alpha)$

Solution

(i) L.H.S = $\sin(45 + \alpha)$

$$= \sin 45^\circ \cos \alpha + \cos 45^\circ \sin \alpha = \left(\frac{1}{\sqrt{2}} \cos \alpha + \frac{1}{\sqrt{2}} \sin \alpha \right)$$

$$= \frac{1}{\sqrt{2}}(\cos \alpha + \sin \alpha) = \frac{1}{\sqrt{2}}(\sin \alpha + \cos \alpha) = \text{R.H.S}$$

(ii) *Do yourself as above*

Question # 4

Prove that:

(i) $\tan(45 + A) \tan(45 - A) = 1$

(ii) $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right) = 0$

(iii) $\sin\left(\theta + \frac{\pi}{6}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos \theta$

(iv) $\frac{\sin \theta - \cos \theta \tan \frac{\theta}{2}}{\cos \theta + \sin \theta \tan \frac{\theta}{2}} = \tan \frac{\theta}{2}$

(v) $\frac{1 - \tan \theta \tan \varphi}{1 + \tan \theta \tan \varphi} = \frac{\cos(\theta + \varphi)}{\cos(\theta - \varphi)}$

Solution

(i) L.H.S = $\tan(45 + A) \tan(45 - A)$

$$= \left(\frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} \right) \left(\frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A} \right)$$

$$= \left(\frac{1 + \tan A}{1 - (1) \tan A} \right) \left(\frac{1 - \tan A}{1 + (1) \tan A} \right) = \left(\frac{1 + \tan A}{1 - \tan A} \right) \left(\frac{1 - \tan A}{1 + \tan A} \right) = 1 = \text{R.H.S}$$

(ii) L.H.S = $\tan\left(\frac{\pi}{4} - \theta\right) + \tan\left(\frac{3\pi}{4} + \theta\right)$

$$= \left(\frac{\tan \frac{\pi}{4} - \tan \theta}{1 + \tan \frac{\pi}{4} \tan \theta} \right) + \left(\frac{\tan \frac{3\pi}{4} + \tan \theta}{1 - \tan \frac{3\pi}{4} \tan \theta} \right)$$

$$\begin{aligned}
&= \left(\frac{1 - \tan \theta}{1 + (1) \tan \theta} \right) + \left(\frac{-1 + \tan \theta}{1 - (-1) \tan \theta} \right) \\
&= \left(\frac{1 - \tan \theta}{1 + \tan \theta} \right) + \left(\frac{-1 + \tan \theta}{1 + \tan \theta} \right) \\
&= \frac{1 - \tan \theta - 1 + \tan \theta}{1 + \tan \theta} = \frac{0}{1 + \tan \theta} = 0 = \text{R.H.S}
\end{aligned}$$

$$\begin{aligned}
\text{(iii) L.H.S} &= \sin \left(\theta + \frac{\pi}{6} \right) + \cos \left(\theta + \frac{\pi}{3} \right) \\
&= \left(\sin \theta \cos \frac{\pi}{6} + \cos \theta \sin \frac{\pi}{6} \right) + \left(\cos \theta \cos \frac{\pi}{3} - \sin \theta \sin \frac{\pi}{3} \right) \\
&= \left(\sin \theta \frac{\sqrt{3}}{2} + \cos \theta \frac{1}{2} \right) + \left(\cos \theta \frac{1}{2} - \sin \theta \frac{\sqrt{3}}{2} \right) \\
&= \frac{\sqrt{3}}{2} \sin \theta + \frac{1}{2} \cos \theta + \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = \cos \theta = \text{R.H.S.}
\end{aligned}$$

$$\begin{aligned}
\text{(iv) L.H.S} &= \frac{\sin \theta - \cos \theta \tan \frac{\theta}{2}}{\cos \theta + \sin \theta \tan \frac{\theta}{2}} \\
&= \frac{\sin \theta - \cos \theta \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}}{\cos \theta + \sin \theta \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}} = \frac{\frac{\sin \theta \cos \frac{\theta}{2} - \cos \theta \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}}{\frac{\cos \theta \cos \frac{\theta}{2} + \sin \theta \sin \frac{\theta}{2}}{\cos \frac{\theta}{2}}} \\
&= \frac{\sin \theta \cos \frac{\theta}{2} - \cos \theta \sin \frac{\theta}{2}}{\cos \theta \cos \frac{\theta}{2} + \sin \theta \sin \frac{\theta}{2}} = \frac{\sin \left(\theta - \frac{\theta}{2} \right)}{\cos \left(\theta - \frac{\theta}{2} \right)} = \frac{\sin \left(\frac{\theta}{2} \right)}{\cos \left(\frac{\theta}{2} \right)} = \tan \frac{\theta}{2} = \text{R.H.S}
\end{aligned}$$

$$\begin{aligned}
\text{(v) L.H.S} &= \frac{1 - \tan \theta \tan \varphi}{1 + \tan \theta \tan \varphi} \\
&= \frac{1 - \frac{\sin \theta}{\cos \theta} \frac{\sin \varphi}{\cos \varphi}}{1 + \frac{\sin \theta}{\cos \theta} \frac{\sin \varphi}{\cos \varphi}} = \frac{\frac{\cos \theta \cos \varphi - \sin \theta \sin \varphi}{\cos \theta \cos \varphi}}{\frac{\cos \theta \cos \varphi + \sin \theta \sin \varphi}{\cos \theta \cos \varphi}} \\
&= \frac{\cos \theta \cos \varphi - \sin \theta \sin \varphi}{\cos \theta \cos \varphi + \sin \theta \sin \varphi} = \frac{\cos(\theta + \varphi)}{\cos(\theta - \varphi)} = \text{R.H.S}
\end{aligned}$$

Question # 5

Show that: $\cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha$

Solution

$$\begin{aligned} \text{L.H.S} &= \cos(\alpha + \beta) \cos(\alpha - \beta) \\ &= (\cos \alpha \cos \beta - \sin \alpha \sin \beta)(\cos \alpha \cos \beta + \sin \alpha \sin \beta) \\ &= ((\cos \alpha \cos \beta)^2 - (\sin \alpha \sin \beta)^2) = \cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta \\ &= \cos^2 \alpha (1 - \sin^2 \beta) - (1 - \cos^2 \alpha) \sin^2 \beta \\ &= \cos^2 \alpha - \cos^2 \alpha \sin^2 \beta - \sin^2 \beta + \cos^2 \alpha \sin^2 \beta \\ &= \cos^2 \alpha - \sin^2 \beta \dots\dots\dots (i) \\ &= (1 - \sin^2 \alpha) - (1 - \cos^2 \beta) \\ &= 1 - \sin^2 \alpha - 1 + \cos^2 \beta \\ &= \cos^2 \beta - \sin^2 \alpha \dots\dots\dots (ii) \end{aligned}$$

Question # 6

*Do yourself as above
Hint: Just open the formulas*

Question # 7

Show that

$$\begin{aligned} \text{(i)} \quad \cot(\alpha + \beta) &= \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta} & \text{(ii)} \quad \cot(\alpha - \beta) &= \frac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha} \\ \text{(iii)} \quad \frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} &= \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} \end{aligned}$$

Solution

$$\begin{aligned} \text{(i)} \quad \text{L.H.S} &= \cot(\alpha + \beta) = \frac{1}{\tan(\alpha + \beta)} = \frac{1}{\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}} \\ &= \frac{1 - \tan \alpha \tan \beta}{\tan \alpha + \tan \beta} = \frac{\tan \alpha \tan \beta \left(\frac{1}{\tan \alpha \tan \beta} - 1 \right)}{\tan \alpha \tan \beta \left(\frac{1}{\tan \beta} + \frac{1}{\tan \alpha} \right)} \\ &= \frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha} = \frac{\cot \alpha \cot \beta - 1}{\cot \alpha + \cot \beta} = \text{R.H.S} \end{aligned}$$

(ii)

Do yourself as above

(iii)

$$\text{L.H.S} = \frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta} = \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{\frac{\sin \alpha}{\cos \alpha} - \frac{\sin \beta}{\cos \beta}} = \frac{\frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}{\frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta}}$$

$$= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \cos \beta - \cos \alpha \sin \beta} = \frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \text{R.H.S}$$

Question # 8

If $\sin \alpha = \frac{4}{5}$ and $\cos \alpha = \frac{40}{41}$ where $0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$.

Show that $\sin(\alpha - \beta) = \frac{133}{205}$

Solution Since $\sin \alpha = \frac{4}{5}$; $0 < \alpha < \frac{\pi}{2}$

$$\cos \alpha = \frac{40}{41} ; \quad 0 < \beta < \frac{\pi}{2}$$

Now

$$\begin{aligned} \cos^2 \alpha &= 1 - \sin^2 \alpha \\ \Rightarrow \cos \alpha &= \pm \sqrt{1 - \sin^2 \alpha} \end{aligned}$$

Since terminal ray of α is in the first quadrant so value of \cos is +ive

$$\begin{aligned} \cos \alpha &= \sqrt{1 - \sin^2 \alpha} \\ \Rightarrow \cos \alpha &= \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} \Rightarrow \boxed{\cos \alpha = \frac{3}{5}} \end{aligned}$$

Also

$$\sin^2 \beta = 1 - \cos^2 \beta \Rightarrow \sin \beta = \pm \sqrt{1 - \cos^2 \beta}$$

Since terminal ray of β is in the first quadrant so value of \sin is +ive

$$\begin{aligned} \sin \beta &= \sqrt{1 - \cos^2 \beta} \\ \Rightarrow \sin \beta &= \sqrt{1 - \left(\frac{40}{41}\right)^2} = \sqrt{1 - \frac{1600}{1681}} = \sqrt{\frac{81}{1681}} \Rightarrow \boxed{\sin \beta = \frac{9}{41}} \end{aligned}$$

Now

$$\begin{aligned} \sin(\alpha - \beta) &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= \left(\frac{4}{5}\right)\left(\frac{40}{41}\right) - \left(\frac{3}{5}\right)\left(\frac{9}{41}\right) = \frac{160}{205} - \frac{27}{205} = \frac{133}{205} \end{aligned}$$

$$\text{i.e. } \sin(\alpha - \beta) = \frac{133}{205} \quad \text{Proved}$$

Question # 9

If $\sin \alpha = \frac{4}{5}$ and $\sin \beta = \frac{12}{13}$ where $\frac{\pi}{2} < \alpha < \pi$ and $\frac{\pi}{2} < \beta < \pi$. Find

- (i) $\sin(\alpha + \beta)$ (ii) $\cos(\alpha + \beta)$ (iii) $\tan(\alpha + \beta)$
 (iv) $\sin(\alpha - \beta)$ (v) $\cos(\alpha - \beta)$ (vi) $\tan(\alpha - \beta)$

In which quadrant do the terminal sides of the angles of measures $(\alpha + \beta)$ and $(\alpha - \beta)$ lie?

Solution

$$\text{Since } \sin \alpha = \frac{4}{5} ; \quad \frac{\pi}{2} < \alpha < \pi$$

$$\sin \beta = \frac{12}{13} ; \quad \frac{\pi}{2} < \beta < \pi$$

$$\text{Since } \cos^2 \alpha = 1 - \sin^2 \alpha \Rightarrow \cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}$$

As terminal ray of α lies in the IInd quadrant so value of cos is -ive

$$\cos \alpha = -\sqrt{1 - \sin^2 \alpha}$$

$$\Rightarrow \cos \alpha = -\sqrt{1 - \left(\frac{4}{5}\right)^2} = -\sqrt{1 - \frac{16}{25}} = -\sqrt{\frac{9}{25}} \Rightarrow \boxed{\cos \alpha = -\frac{3}{5}}$$

Now

$$\cos^2 \beta = 1 - \sin^2 \beta$$

$$\Rightarrow \cos \beta = \pm \sqrt{1 - \sin^2 \beta}$$

As terminal ray of β lies in the IInd quadrant so value of cos is -ive

$$\cos \beta = -\sqrt{1 - \sin^2 \beta}$$

$$\Rightarrow \cos \beta = -\sqrt{1 - \left(\frac{12}{13}\right)^2} = -\sqrt{1 - \frac{144}{169}} = -\sqrt{\frac{25}{169}} \Rightarrow \boxed{\cos \beta = -\frac{5}{13}}$$

$$(i) \quad \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$= \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right) + \left(-\frac{3}{5}\right)\left(\frac{12}{13}\right) = -\frac{20}{65} - \frac{36}{65} = -\frac{56}{65}$$

$$(ii) \quad \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$= \left(-\frac{3}{5}\right)\left(-\frac{5}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{12}{13}\right) = \frac{15}{65} - \frac{48}{65} = -\frac{33}{65}$$

$$(iii) \quad \tan(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} = \frac{-56/65}{-33/65} = \frac{56}{33}$$

(iv), (v) & (vi) *Do yourself as above*

Since $\sin(\alpha + \beta)$ is -ive so terminal arc of $\alpha + \beta$ is in IIIrd or IVth quadrant and $\cos(\alpha + \beta)$ is -ive so terminal arc of $\alpha + \beta$ is in IInd or IIIrd quadrant therefore terminal ray of $\alpha + \beta$ lies in the IIIrd quadrant.

Similarly after solving (iv), (v) & (vi) find quadrant for $\alpha - \beta$ yourself.

Question # 10

Find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, given that

(i) $\tan \alpha = \frac{3}{4}$, $\sin \beta = \frac{5}{13}$ and neither the terminal side of the angle of measure α nor that of β is in the I quadrant.

(ii) $\tan \alpha = -\frac{15}{8}$, $\sin \beta = -\frac{7}{25}$ and neither the terminal side of the angle of measure α nor that of β is in the IV quadrant.

Solution

(i) Since $\tan \alpha = \frac{3}{4}$

As $\tan \alpha$ is +ive and terminal arm of α is not in the Ist quad. Therefore it lies in IIIrd quad.

Now

$$\sec^2 \alpha = 1 + \tan^2 \alpha$$

$$\Rightarrow \sec \alpha = \pm \sqrt{1 + \tan^2 \alpha}$$

Since terminal arm of α is in the IIIrd quad., therefore value of sec is -ive

$$\sec \alpha = -\sqrt{1 + \tan^2 \alpha}$$

$$\Rightarrow \sec \alpha = -\sqrt{1 + \left(\frac{3}{4}\right)^2} = -\sqrt{1 + \frac{9}{16}} = -\sqrt{\frac{25}{16}} \Rightarrow \sec \alpha = -\frac{5}{4}$$

Now $\cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{-\frac{5}{4}} \Rightarrow \boxed{\cos \alpha = -\frac{4}{5}}$

Now $\frac{\sin \alpha}{\cos \alpha} = \tan \alpha \Rightarrow \sin \alpha = \tan \alpha \cos \alpha$

$$\Rightarrow \sin \alpha = \left(\frac{3}{4}\right)\left(-\frac{4}{5}\right) \Rightarrow \boxed{\sin \alpha = -\frac{3}{5}}$$

Since $\cos \beta = \frac{5}{13}$

As $\cos \beta$ is +ive and terminal arm of β is not in the Ist quad., therefore it lies in IVth quad.

Now $\sin^2 \beta = 1 - \cos^2 \beta$

$$\Rightarrow \sin \beta = \pm \sqrt{1 - \cos^2 \beta}$$

Since terminal ray of β is in the fourth quadrant so value of sin is -ive

$$\sin \beta = -\sqrt{1 - \cos^2 \beta}$$

$$\Rightarrow \sin \beta = -\sqrt{1 - \left(\frac{5}{13}\right)^2} = -\sqrt{1 - \frac{25}{169}} = -\sqrt{\frac{144}{169}} \Rightarrow \boxed{\sin \beta = -\frac{12}{13}}$$

Now $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \left(-\frac{3}{5}\right)\left(\frac{5}{13}\right) + \left(-\frac{4}{5}\right)\left(-\frac{12}{13}\right) = -\frac{3}{13} + \frac{48}{65} \Rightarrow \boxed{\sin(\alpha + \beta) = \frac{33}{65}}$$

And $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

$$= \left(-\frac{4}{5}\right)\left(\frac{5}{13}\right) - \left(-\frac{3}{5}\right)\left(-\frac{12}{13}\right) = -\frac{4}{13} - \frac{36}{65} \Rightarrow \boxed{\cos(\alpha + \beta) = -\frac{56}{65}}$$

(ii)

Do yourself as above

Question # 11

Prove that: $\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \tan 37^\circ$

Solution

$$\begin{aligned} \text{R.H.S} &= \tan 37^\circ = \tan(45 - 8) && \because 37 = 45 - 8 \\ &= \frac{\tan 45^\circ - \tan 8^\circ}{1 + \tan 45^\circ \tan 8^\circ} = \frac{1 - \tan 8^\circ}{1 + (1)\tan 8^\circ} \\ &= \frac{1 - \frac{\sin 8^\circ}{\cos 8^\circ}}{1 + \frac{\sin 8^\circ}{\cos 8^\circ}} = \frac{\frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ}}{\frac{\cos 8^\circ + \sin 8^\circ}{\cos 8^\circ}} = \frac{\cos 8^\circ - \sin 8^\circ}{\cos 8^\circ + \sin 8^\circ} = \text{L.H.S} \end{aligned}$$

Question # 12

If α, β, γ are the angles of a triangle ABC , show that

$$\cot \frac{\beta}{2} + \cot \frac{\alpha}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

Solution

Since α, β and γ are angles of triangle therefore

$$\begin{aligned} \alpha + \beta + \gamma &= 180 \Rightarrow \alpha + \beta = 180 - \gamma \\ \Rightarrow \frac{\alpha + \beta}{2} &= \frac{180 - \gamma}{2} \Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} = 90 - \frac{\gamma}{2} \end{aligned}$$

Now $\tan\left(\frac{\alpha}{2} + \frac{\beta}{2}\right) = \tan\left(90 - \frac{\gamma}{2}\right)$

$$\Rightarrow \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = \cot \frac{\gamma}{2} \qquad \because \tan\left(90 - \frac{\gamma}{2}\right) = \cot \frac{\gamma}{2}$$

$$\Rightarrow \frac{\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \left(\frac{1}{\tan \frac{\beta}{2}} + \frac{1}{\tan \frac{\alpha}{2}}\right)}{\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \left(\frac{1}{\tan \frac{\alpha}{2} \tan \frac{\beta}{2}} - 1\right)} = \cot \frac{\gamma}{2} \qquad \Rightarrow \frac{\cot \frac{\beta}{2} + \cot \frac{\alpha}{2}}{\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - 1} = \cot \frac{\gamma}{2}$$

$$\Rightarrow \cot \frac{\beta}{2} + \cot \frac{\alpha}{2} = \cot \frac{\gamma}{2} \left(\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - 1\right)$$

$$\Rightarrow \cot \frac{\beta}{2} + \cot \frac{\alpha}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2} - \cot \frac{\gamma}{2}$$

$$\Rightarrow \cot \frac{\beta}{2} + \cot \frac{\alpha}{2} + \cot \frac{\gamma}{2} = \cot \frac{\alpha}{2} \cot \frac{\beta}{2} \cot \frac{\gamma}{2}$$

Question # 13

If $\alpha + \beta + \gamma = 180^\circ$, show that
 $\cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha = 1$

Solution

Since α, β and γ are angles of triangle therefore

$$\alpha + \beta + \gamma = 180 \quad \Rightarrow \quad \alpha + \beta = 180 - \gamma$$

Now $\tan(\alpha + \beta) = \tan(180 - \gamma)$

$$\Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \tan(2(90) - \gamma)$$

$$\Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = -\tan \gamma$$

$$\Rightarrow \tan \alpha + \tan \beta = -\tan \gamma(1 - \tan \alpha \tan \beta)$$

$$\Rightarrow \tan \alpha + \tan \beta = -\tan \gamma + \tan \alpha \tan \beta \tan \gamma$$

$$\Rightarrow \tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$$

Dividing through out by $\tan \alpha \tan \beta \tan \gamma$

$$\frac{\tan \alpha}{\tan \alpha \tan \beta \tan \gamma} + \frac{\tan \beta}{\tan \alpha \tan \beta \tan \gamma} + \frac{\tan \gamma}{\tan \alpha \tan \beta \tan \gamma} = \frac{\tan \alpha \tan \beta \tan \gamma}{\tan \alpha \tan \beta \tan \gamma}$$

$$\Rightarrow \cot \beta \cot \gamma + \cot \alpha \cot \gamma + \cot \alpha \cot \beta = 1$$

$$\Rightarrow \cot \alpha \cot \beta + \cot \beta \cot \gamma + \cot \gamma \cot \alpha = 1$$

Question # 14

Express the following in the form $r \sin(\theta + \phi)$ or $r \sin(\theta - \phi)$, where terminal sides of the angles of measure θ and ϕ are in the first quadrant:

- (i) $12 \sin \theta + 5 \cos \theta$ (ii) $3 \sin \theta - 4 \cos \theta$ (iii) $\sin \theta - \cos \theta$
 (iv) $5 \sin \theta - 4 \cos \theta$ (v) $\sin \theta + \cos \theta$

Solution

(i) $12 \sin \theta + 5 \cos \theta$

Let $12 = r \cos \phi$ and $5 = r \sin \phi$

Squaring and adding

$$(12)^2 + (5)^2 = r^2 \cos^2 \phi + r^2 \sin^2 \phi$$

$$\Rightarrow 144 + 25 = r^2 (\cos^2 \phi + \sin^2 \phi)$$

$$\Rightarrow 169 = r^2 (1)$$

$$\Rightarrow r = \sqrt{169} = 13$$

Now

$$12 \sin \theta + 5 \cos \theta = r \cos \phi \sin \theta + r \sin \phi \cos \theta$$

$$= r (\cos \phi \sin \theta + \sin \phi \cos \theta)$$

$$\frac{5}{12} = \frac{r \sin \phi}{r \cos \phi}$$

$$\frac{5}{12} = \tan \phi$$

$$\Rightarrow \phi = \tan^{-1} \frac{5}{12}$$

$$= r \sin(\theta + \varphi)$$

where $r = 13$ and $\varphi = \tan^{-1} \frac{5}{12}$

(ii) $3 \sin \theta - 4 \cos \theta$

Let $3 = r \cos \varphi$ and $-4 = r \sin \varphi$

Squaring and adding

$$(3)^2 + (-4)^2 = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi$$

$$\Rightarrow 9 + 16 = r^2 (\cos^2 \varphi + \sin^2 \varphi)$$

$$\Rightarrow 25 = r^2 (1)$$

$$\Rightarrow r = \sqrt{25} = 5$$

$$\frac{-4}{3} = \frac{r \sin \varphi}{r \cos \varphi}$$

$$-\frac{4}{3} = \tan \varphi$$

$$\Rightarrow \varphi = \tan^{-1} \left(-\frac{4}{3} \right)$$

$$3 \sin \theta - 4 \cos \theta = r \cos \varphi \sin \theta + r \sin \varphi \cos \theta$$

$$= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta)$$

$$= r \sin(\theta + \varphi)$$

where $r = 5$ and $\varphi = \tan^{-1} \left(-\frac{4}{3} \right)$

(iii) $\sin \theta - \cos \theta$

Let $1 = r \cos \varphi$ and $-1 = r \sin \varphi$

Squaring and adding

$$(1)^2 + (-1)^2 = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi$$

$$\Rightarrow 1 + 1 = r^2 (\cos^2 \varphi + \sin^2 \varphi)$$

$$\Rightarrow 2 = r^2 (1)$$

$$\Rightarrow r = \sqrt{2}$$

$$\frac{-1}{1} = \frac{r \sin \varphi}{r \cos \varphi}$$

$$-1 = \tan \varphi$$

$$\Rightarrow \varphi = \tan^{-1} (-1)$$

Now

$$\sin \theta - \cos \theta = r \cos \varphi \sin \theta + r \sin \varphi \cos \theta$$

$$= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta)$$

$$= r \sin(\theta + \varphi)$$

where $r = \sqrt{2}$ and $\varphi = \tan^{-1} (-1)$

(iv) $5 \sin \theta - 4 \cos \theta$

Let $5 = r \cos \varphi$ and $-4 = r \sin \varphi$

Squaring and adding

$$(5)^2 + (-4)^2 = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi$$

$$\Rightarrow 25 + 16 = r^2 (\cos^2 \varphi + \sin^2 \varphi)$$

$$\Rightarrow 41 = r^2 (1)$$

$$\Rightarrow r = \sqrt{41}$$

$$\frac{-4}{5} = \frac{r \sin \varphi}{r \cos \varphi}$$

$$-\frac{4}{5} = \tan \varphi$$

$$\Rightarrow \varphi = \tan^{-1} \left(-\frac{4}{5} \right)$$

Now

$$5 \sin \theta - 4 \cos \theta = r \cos \varphi \sin \theta + r \sin \varphi \cos \theta$$

$$= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta)$$

$$= r \sin(\theta + \varphi) \quad \text{where } r = \sqrt{41} \quad \text{and} \quad \varphi = \tan^{-1}\left(-\frac{4}{5}\right)$$

(v) $\sin \theta + \cos \theta$

Let $1 = r \cos \varphi$ and $1 = r \sin \varphi$

Squaring and adding

$$(1)^2 + (1)^2 = r^2 \cos^2 \varphi + r^2 \sin^2 \varphi$$

$$\Rightarrow 1+1 = r^2 (\cos^2 \varphi + \sin^2 \varphi)$$

$$\Rightarrow 2 = r^2 (1)$$

$$\Rightarrow r = \sqrt{2}$$

$$\left. \begin{array}{l} \frac{1}{1} = \frac{r \sin \varphi}{r \cos \varphi} \\ 1 = \tan \varphi \\ \Rightarrow \varphi = \tan^{-1}(1) \end{array} \right\}$$

Now $\sin \theta + \cos \theta = r \cos \varphi \sin \theta + r \sin \varphi \cos \theta$

$$= r (\cos \varphi \sin \theta + \sin \varphi \cos \theta)$$

$$= r \sin(\theta + \varphi) \quad \text{where } r = \sqrt{2} \quad \text{and} \quad \varphi = \tan^{-1}(1)$$

(vi) *Do yourself as above*

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Book: *Exercise 10.2 (Page 326)*
Text Book of Algebra and Trigonometry Class XI
Punjab Textbook Board, Lahore.

Available online at <http://www.MathCity.org> in PDF Format

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